

SIMPLE LIE ALGEBRAS OF SMALL CHARACTERISTIC VI. COMPLETION OF THE CLASSIFICATION

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ABSTRACT. Let L be a finite-dimensional simple Lie algebra over an algebraically closed field of characteristic $p > 3$. It is proved in this paper that if the p -envelope of $\text{ad } L$ in $\text{Der } L$ contains a torus of maximal dimension whose centralizer in $\text{ad } L$ acts nontriangulably on L , then $p = 5$ and L is isomorphic to one of the Melikian algebras $\mathcal{M}(m, n)$. In conjunction with [P-St 05, Thm. 1.2], this implies that, up to isomorphism, any finite-dimensional simple Lie algebra over an algebraically closed field of characteristic $p > 3$ is either classical or a filtered Lie algebra of Cartan type or a Melikian algebra of characteristic 5. This result finally settles the classification problem for finite-dimensional simple Lie algebras over algebraically closed fields of characteristic $\neq 2, 3$.

1. Introduction

This paper concludes the series [P-St 97], [P-St 99], [P-St 01], [P-St 04], [P-St 05]. Its goal is to finish the proof of the following theorem which was announced in [St 04] and [P-St 06]:

Theorem 1.1 (Classification Theorem). *Any finite-dimensional simple Lie algebra over an algebraically closed field of characteristic $p > 3$ is of classical, Cartan or Melikian type.*

For $p > 7$, the finite-dimensional simple Lie algebras were classified by the second author in the series of papers [St 89], [St 91], [St 92], [St 93], [St 94], [St 98]. It should be mentioned that the Classification Theory was inspired by the ground-breaking work of Block–Wilson [B-W 82], [B-W 88] who handled the so-called restricted case (also for $p > 7$).

In what follows, F will denote an algebraically closed field of characteristic $p > 3$, and L will always stand for a finite-dimensional simple Lie algebra over F . As usual, we identify L with the subalgebra $\text{ad } L$ of the derivation algebra $\text{Der } L$ and denote by L_p the semisimple p -envelope of L (it coincides with the p -closure of $\text{ad } L$ in the restricted Lie algebra $\text{Der } L$). Given a torus T of maximal dimension in L_p we let H stand for the centralizer of T in L ; that is,

$$H := \mathfrak{c}_L(T) = \{x \in L \mid [t, x] = 0 \ \forall t \in T\}.$$

Let $\Gamma(L, T)$ be the set of roots of L relative to T ; that is, the set of all *nonzero* linear functions $\gamma \in T^*$ for which the subspace $L_\gamma := \{x \in L \mid [t, x] = \gamma(t)x \ \forall t \in T\}$ is nonzero. Then H is a nilpotent subalgebra of L (possibly zero) and L decomposes as $L = H \oplus \bigoplus_{\gamma \in \Gamma(L, T)} L_\gamma$. By [P-St 04, Cor. 3.7] any root γ in $\Gamma(L, T)$ is either *solvable* or *classical* or *Witt* or *Hamiltonian*. Accordingly, the semisimple quotient

$L[\gamma] = L(\gamma)/\text{rad } L(\gamma)$ of the 1-section $L(\gamma) := H \oplus \bigoplus_{i \in \mathbb{F}_p^\times} L_{i\gamma}$ is either (0) or $\mathfrak{sl}(2)$ or the Witt algebra $W(1; \underline{1})$ or contains an isomorphic copy of the Hamiltonian algebra $H(2; \underline{1})^{(2)}$ as an ideal of codimension ≤ 1 . For $\alpha, \beta \in \Gamma(L, T)$ we denote by $L(\alpha, \beta)$ the 2-section $\sum_{i, j \in \mathbb{F}_p} L_{i\alpha + j\beta}$, where $L_0 = H$ by convention.

We say that T is *standard* if $H^{(1)}$ consists of nilpotent derivations of L and *nonstandard* otherwise. In [P-St 04] and [P-St 05], it was shown that if all tori of maximal dimension in L_p are standard, then L is either classical or a filtered Lie algebra of Cartan type. On the other hand, the main results of [P 94] imply that if L_p contains a nonstandard torus of maximal dimension, say T' , then there are $\alpha, \beta \in \Gamma(L, T')$ such that the factor algebra $L(\alpha, \beta)/\text{rad } L(\alpha, \beta)$ is isomorphic to the restricted Melikian algebra $\mathcal{M}(1, 1)$. In particular, $p = 5$ in this case.

The main result of the present paper is the following:

Theorem 1.2. *If the semisimple p -envelope of L contains nonstandard tori of maximal dimension, then L is isomorphic to one of the Melikian algebras $\mathcal{M}(m, n)$, where $(m, n) \in \mathbb{N}^2$.*

Together with the main results of [P-St 04] and [P-St 05] Theorem 1.2 implies the Classification Theorem. In view of [St 04, Cor. 7.2.3] we also obtain:

Corollary 1.3. *Any finite-dimensional restricted simple Lie algebra over an algebraically closed field of characteristic $p > 3$ is, up to isomorphism, either one of $W(n; \underline{1})$, $n \geq 1$, $S(n; \underline{1})^{(1)}$, $n \geq 3$, $H(2r; \underline{1})^{(2)}$, $r \geq 1$, $K(2r + 1; \underline{1})^{(1)}$, $r \geq 1$, $\mathcal{M}(1, 1)$, or has the form $(\text{Lie } G)^{(1)}$, where G is a simple algebraic F -group of adjoint type.*

For the reader's convenience, we now give a brief overview of the proof of Theorem 1.2. Since our goal is to show that $L \cong \mathcal{M}(m, n)$, we need to produce a subalgebra $L_{(0)}$ of codimension 5 in L . As in the previous two papers of the series, local analysis is vital here. All possible types of 2-sections in simple Lie algebras are described in [P-St 04, Sect. 4]. The list of 2-sections is long, but a thorough investigation shows that most of them cannot occur in our situation. We prove in Section 5 that if T is a nonstandard torus of maximal dimension in L_p and $\alpha, \beta \in \Gamma(L, T)$ are \mathbb{F}_p -independent, then $\text{rad } L(\alpha, \beta) \subset T$ and either $L[\alpha, \beta] \cong \mathcal{M}(1, 1)$ of $L[\alpha, \beta]^{(1)} \cong H(2; (2, 1))^{(2)}$; see Theorem 5.8. In particular, this implies that all root spaces of L with respect to T are 5-dimensional. This intermediate result is crucial for the rest of the paper. In order to prove it we have to refine our earlier description of 2-sections with core of type $H(2; (2, 1))^{(2)}$; see Theorem 3.6(5). The proof of Theorem 3.6(5) relies heavily on a classification of certain toral derivations of $H(2; (2, 1))$. The latter is obtained in Section 2, the longest section of the paper.

In Section 6, we show the restricted Melikian algebra $\mathcal{M}(1, 1)$ has no nontrivial central extensions and describe the p -characters of irreducible $\mathcal{M}(1, 1)$ -modules of dimension ≤ 125 . This gives us important new information on the p -mapping of L_p ; see Section 7. To proceed further we need a sufficiently generic nonstandard torus of maximal dimension in L_p . We show in Section 9 that there is a nonstandard torus T of maximal dimension in L_p for which $H^3 = [\mathbf{c}_L(T), [\mathbf{c}_L(T), \mathbf{c}_L(T)]]$ contains no nonzero toral elements. We then use the new information on the p -mapping of L_p to construct for every $\alpha \in \Gamma(L, T)$ a subalgebra $Q(\alpha) \subset L(\alpha)$ such that $L(\alpha) = H \oplus Q(\alpha)$, and set $L_{(0)} := \sum_{\alpha \in \Gamma(L, T)} Q(\alpha)$. By construction, $L_{(0)}$ is a subspace of L . In order to show

that it is a subalgebra, we need to check that $[Q(\alpha), Q(\beta)] \subset Q(\alpha) \oplus \sum_{i \in \mathbb{F}_p} Q(\beta + i\alpha)$ for all \mathbb{F}_p -independent $\alpha, \beta \in \Gamma(L, T)$. This is carried out in Section 10. The rest of the proof is routine.

All Lie algebras in this paper are assumed to be finite-dimensional. We adopt the notation introduced in [P-St 97], [P-St 99], [P-St 01], [P-St 04] with the following two exceptions: the divided power algebra $A(m; \underline{n})$ is denoted here by $\mathcal{O}(m; \underline{n})$, and the Melikian algebra $\mathfrak{g}(m, n)$ by $\mathcal{M}(m, n)$. Given a Lie subalgebra M of L , we write M_p for the p -envelope of M in L_p .

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2. Toral elements and one-sections in $H(2; (2, 1))$

The Lie algebra $H(2; (2, 1))$ will appear quite frequently in what follows, and to deal with it we need some refinements of [B-W 88, (10.1.1)], [St 91, (VI.4)] and [P-St 04, Prop. 2.1]. Set $S := H(2; (2, 1))^{(2)}$, $G := H(2; (2, 1))$, and denote by $S_{(i)}$ (resp., $G_{(i)}$) the i th component of the standard filtration of S (resp., G). Recall that $S_p = H(2; (2, 1))^{(2)} \oplus FD_1^p$; see [St 04, Thm. 7.2.2(5)], for instance. By [B-W 88, Prop. 2.1.8(viii)], $G = V \oplus S$ where

$$V = FD_H(x_1^{(p^2)}) \oplus FD_H(x_2^{(p)}) \oplus D_H(x_1^{(p^2-1)}x_2^{(p-1)}).$$

Note that V is a Lie subalgebra of G , and in $\text{Der } S$ we have $V^{[p]} = V^3 = 0$. We denote by \mathcal{G} the p -envelope of G in $\text{Der } S$. As $V^{[p]} = 0$, it follows from Jacobson's formula [St 04, p. 17] that $\mathcal{G} = V \oplus S_p$. We remind the reader that G is a Lie subalgebra of the Hamiltonian algebra $H(2) = \text{span}\{D_H(f) \mid f \in \mathcal{O}(2)\}$ and

$$[D_H(f), D_H(g)] = D_H(D_1(f)D_2(g) - D_2(f)D_1(g)) \quad (\forall f, g \in \mathcal{O}(2)).$$

Furthermore, $D_H(f) = D_H(g)$ if and only if $f - g \in F$.

Lemma 2.1. *Every toral element t of S_p contained in $S \setminus S_{(0)}$ is conjugate under the automorphism group of S to an element*

$$t_\mu = D_H(x_1 + \mu x_1^{(p)} + (x_1 + \mu x_1^{(p)})r x_2^{(p-1)}), \quad r = 1 + \mu x_1^{(p-1)},$$

where $\mu \in \{0, 1\}$. Each such element is toral.

Proof. (a) Write $t = aD_1 + bD_2 + w$ with $a, b \in F$ and $w \in S_{(0)}$. By our assumption, t is a toral element of S_p ; that is, $t^{[p]} = t$. Since $(aD_1 + bD_2)^{[p]} = a^p D_1^p$ and $w^{[p]} \in S_{(0)}$, Jacobson's formula yields $a = 0$. Since $t \notin S_{(0)}$, it must be that $b \neq 0$. There exists a special automorphism σ of the divided power algebra $\mathcal{O}(2; (2, 1))$ such that $\sigma(x_1) = b^{-1}x_1$ and $\sigma(x_2) = bx_2$. It induces an automorphism Φ_σ of the Lie algebra S via $\Phi_\sigma(E) = \sigma \circ E \circ \sigma^{-1}$ for all $E \in S$; see [St 04, Thm. 7.3.6]. After adjusting t by Φ_σ it can be assumed that $b = 1$. The description of $\text{Aut } S$ given in [St 04, Thms 7.3.5 & 7.3.2] implies that for any $\lambda \in F$ and any pair of nonnegative integers (m, n) such that either $(m, n) = (p^2, 0)$ or $m + n \geq 3$, $m < p^2$, $n < p$ and $(m, n) \neq (p, 1)$ there exists $\sigma_{m,n,\lambda} \in \text{Aut } S$ with

$$\sigma_{m,n,\lambda}(u) \equiv u + \lambda [D_H(x_1^{(m)}x_2^{(n)}), u] \pmod{S_{(i+m+n-1)}} \quad (\forall u \in S_{(i)}).$$

Because

$$[D_2, D_H(x_1^{(m)} x_2^{(n)})] = D_H(x_1^{(m)} x_2^{(n-1)}) \quad (1 \leq n \leq p-1),$$

it is not hard to see that there is $g \in \text{Aut } S$ such that $g(t) = D_H(x_1 + \mu x_1^{(p)}) + D_H(f x_2^{(p-1)})$ for some $\mu \in F$ and $f = \sum_{i=1}^{p^2-1} \lambda_i x_1^{(i)}$ with $\lambda_i \in F$. If $\mu \neq 0$, then there exists $\alpha \in F$ with $\alpha^{p-1} \mu = 1$ and a special automorphism σ' of the divided power algebra $\mathcal{O}(2; (2, 1))$ for which $\sigma'(x_1) = \alpha x_1$ and $\sigma'(x_2) = x_2$. It gives rise to an automorphism $\Phi_{\sigma'}$ of the Lie algebra S such that $\Phi_{\sigma'}(D_H(x_1^{(r)} x_2^{(s)})) = \alpha^{r-1} D_H(x_1^{(r)} x_2^{(s)})$ for all admissible r and s ; see [St 04, Thm. 7.3.6]. Adjusting t by $\Phi_{\sigma'}$ we may assume without loss that $\mu \in \{0, 1\}$.

Put $r = D_1(x_1 + \mu x_1^{(p)}) = 1 + \mu x_1^{(p-1)}$, $f' := D_1(f)$, and assume from now on that $t = D_H(x_1 + \mu x_1^{(p)}) + D_H(f x_2^{(p-1)})$.

(b) As $(\text{ad } D_H(f x_2^{(p-1)}))(\text{ad } D_H(x_1 + \mu x_1^{(p)}))^k (D_H(f x_2^{(p-1)})) = 0$ for $0 \leq k \leq p-3$, $D_H(x_1 + \mu x_1^{(p)})^{[p]} = D_H(f x_2^{(p-1)})^{[p]} = 0$, and

$$[D_H(x_1 + \mu x_1^{(p)}), D_H(r^i f x_2^{(j)})] = D_H(r^{i+1} f x_2^{(j-1)}) \quad (1 \leq i, j \leq p-1),$$

Jacobson's formula yields

$$\begin{aligned} t^{[p]} &= (\text{ad } D_H(x_1 + \mu x_1^{(p)}))^{p-1} (D_H(f x_2^{(p-1)})) \\ &+ \frac{1}{2} [D_H(f x_2^{(p-1)}), (\text{ad } D_H(x_1 + \mu x_1^{(p)}))^{p-2} (D_H(f x_2^{(p-1)}))] \\ &= D_H(r^{p-1} f) + \frac{1}{2} [D_H(f x_2^{(p-1)}), D_H(r^{p-2} f x_2)] \\ &= D_H(r^{p-1} f) + \frac{1}{2} D_H(f' r^{p-2} f x_2^{(p-1)}) - \frac{1}{2} \binom{p-1}{1} D_H(f D_1(r^{p-2} f) x_2^{(p-1)}) \\ &= D_H(r^{p-1} f) + D_H(f f' r^{p-2} x_2^{(p-1)}) - \mu D_H(r^{p-3} x_1^{(p-2)} f^2 x_2^{(p-1)}). \end{aligned}$$

As $r^{p-1} = r^{-1}$, the RHS equals t if and only if $f = (x_1 + \mu x_1^{(p)})r$, as claimed. \square

Denote by $\mathcal{O}(2; (2, 1))_{(k)}[x_1]$ the subalgebra of $\mathcal{O}(2; (2, 1))$ spanned by all $x_1^{(i)}$ with $k \leq i < p^2$ and let $\mathcal{O}(2; (2, 1))[x_1] := \mathcal{O}(2; (2, 1))_{(0)}[x_1]$. For $u \in \mathcal{O}(2; (2, 1))[x_1]$ put $u' := D_1(u)$ and set $\tilde{r} := x_1 + \mu x_1^{(p)}$, so that $t_\mu = D_H(\tilde{r} + r \tilde{r} x_2^{(p-1)})$. Note that $\tilde{r}' = r$.

Lemma 2.2. *Let t_μ be as in Lemma 2.1 and put $C_\mu := \mathfrak{c}_\mathcal{G}(t_\mu)$.*

(i) *The Lie algebra C_μ has an abelian ideal C'_μ of codimension 2 spanned by all $D_H(u + u' \tilde{r} x_2^{(p-1)})$ with $u \in \mathcal{O}(2; (2, 1))[x_1]$ and by $D_H(x_1^{(p^2)})$. Furthermore, $C_\mu = F n_\mu \oplus F h_\mu \oplus C'_\mu$, where $n_\mu = D_1 + \mu D_H(x_2^{(p)})$ and $h_\mu = D_H(r^{-1} x_2 - x_2^{(p)})$.*

(ii) *Given $a \in F$ and $v \in \mathcal{O}(2; (2, 1))[x_1]$ put*

$$\varphi_a(v) := \sum_{i=0}^{p-1} a^i D_H(r^{-i} v x_2^{(i)}) + a^{p-1} D_H(\tilde{r} v' x_2^{(p-1)}).$$

Then for every $k \in \mathbb{F}_p^\times$ the k -eigenspace of $\text{ad } t_\mu$ has dimension p^2 and is spanned by all $\varphi_k(u)$ with $u \in \mathcal{O}(2; (2, 1))[x_1]$.

(iii) *In \mathcal{G} we have $h_\mu^{[p]} = -\mu h_\mu - n_\mu$ and $n_\mu^{[p]} = 0$.*

(iv) If $\mu = 0$, then C_μ is nilpotent and Ft_μ is a maximal torus in \mathcal{G} .

Proof. (i) It is straightforward to see that C'_μ is abelian and $t_\mu \in C'_\mu$. Also,

$$[D_1^p, t_\mu] = \mu D_H(rx_2^{(p-1)}) = -\mu [D_H(x_2^{(p)}), t_\mu],$$

implying $n_\mu \in C_\mu$. For all $u \in \langle x_1^{(i)} \mid 0 \leq i \leq p^2 \rangle$ we have

$$\begin{aligned} [D_H(r^{-1}x_2), D_H(u + u'\tilde{r}x_2^{(p-1)})] &= -D_H(r^{-1}u') + D_H(r^{-2}(r'u'\tilde{r} - (u'\tilde{r})'r)x_2^{(p-1)}) \\ [D_H(x_2^{(p)}), D_H(u + u'\tilde{r}x_2^{(p-1)})] &= -D_H(u'x_2^{(p-1)}). \end{aligned}$$

As a consequence,

$$(2.1) \quad [h_\mu, D_H(u + u'\tilde{r}x_2^{(p-1)})] = -D_H(r^{-1}u' + (r^{-1}u')'\tilde{r}x_2^{(p-1)})$$

for all $u \in \mathcal{O}(2; (2, 1))[x_1]$. Putting $u = \tilde{r}$ gives $h_\mu \in C_\mu$.

(ii) We claim that for all $u \in \langle x_1^{(i)} \mid 1 \leq i \leq p^2 \rangle$ and all $k \in \mathbb{F}_p^\times$ the following relations hold:

$$(2.2) \quad [D_H(u + u'\tilde{r}x_2^{(p-1)}), \varphi_k(v)] = k\varphi_k(r^{-1}u'v)$$

$$(2.3) \quad [D_H(r^{-1}x_2 - x_2^{(p)}), \varphi_k(v)] = [h_\mu, \varphi_k(v)] = -\varphi_k(r^{-1}v').$$

Indeed, since $k^{p-1} = 1$, $r^p = 1$, and $x_2^{(p-2)} \cdot x_2^{(k)} = 0$ for $2 \leq k \leq p-1$, the LHS of (2.2) equals $D_H(w)$, where

$$\begin{aligned} w &= D_1(u + u'\tilde{r}x_2^{(p-1)}) \cdot D_2(\varphi_k(v)) - D_2(u + u'\tilde{r}x_2^{(p-1)}) \cdot D_1(\varphi_k(v)) \\ &= (u' + u''\tilde{r}x_2^{(p-1)} + u'r x_2^{(p-1)}) \cdot \left(\sum_{i=1}^{p-1} k^i r^{-i} v x_2^{(i-1)} + \tilde{r}v'x_2^{(p-2)} \right) \\ &\quad - u'\tilde{r}x_2^{(p-2)} \cdot \left(v' + k(r^{-1}v)'x_2 \right) \\ &= u' \left(\sum_{i=1}^{p-1} k^i r^{-i} v x_2^{(i-1)} \right) + u'\tilde{r}v'x_2^{(p-2)} \\ &\quad + ku''\tilde{r}r^{-1}v x_2^{(p-1)} + ku'v x_2^{(p-1)} - u'\tilde{r}v'x_2^{(p-2)} + ku'\tilde{r}(r^{-1}v)'x_2^{(p-1)} \\ &= k \sum_{i=0}^{p-2} k^i r^{-i} (r^{-1}u'v)x_2^{(i)} + k \left(u'\tilde{r}r^{-1}v + u'v + u'\tilde{r}(r^{-1}v)' \right) x_2^{(p-1)} \\ &= k \sum_{i=0}^{p-1} k^i r^{-i} (r^{-1}u'v)x_2^{(i)} + k\tilde{r}(r^{-1}u'v)'x_2^{(p-1)}. \end{aligned}$$

But then $D_H(w) = k\varphi_k(r^{-1}u'v)$ and (2.2) follows. Since

$$(-r^{-1}v')' = r^{-2}r'v' - r^{-1}v'',$$

the LHS of (2.3) equals $D_H(y)$, where

$$\begin{aligned}
y &= (r^{-1})'x_2 \cdot \left(\sum_{i=1}^{p-1} k^i r^{-i} v x_2^{(i-1)} + \tilde{r} v' x_2^{(p-2)} \right) \\
&- r^{-1} \cdot \left(\sum_{i=1}^{p-1} k^i i (r^{-1})^{i-1} (-r^{-2} r') v x_2^{(i)} + \sum_{i=0}^{p-1} k^i r^{-i} v' x_2^{(i)} \right) \\
&- r^{-1} \cdot (\tilde{r} v')' x_2^{(p-1)} + x_2^{(p-1)} v' \\
&= -r^{-2} r' \cdot \left(\sum_{i=1}^{p-1} k^i i r^{-i} v x_2^{(i)} - \tilde{r} v' x_2^{(p-1)} \right) \\
&+ r^{-2} r' \cdot \left(\sum_{i=1}^{p-1} k^i i r^{-i} v x_2^{(i)} \right) + \sum_{i=0}^{p-1} k^i r^{-i} (-r^{-1} v') x_2^{(i)} \\
&- r^{-1} \cdot (r v' + \tilde{r} v'') x_2^{(p-1)} + x_2^{(p-1)} v' \\
&= r^{-2} r' \tilde{r} v' x_2^{(p-1)} + \sum_{i=0}^{p-1} k^i r^{-i} (-r^{-1} v') x_2^{(i)} - r^{-1} \tilde{r} v'' x_2^{(p-1)} \\
&= \sum_{i=0}^{p-1} k^i r^{-i} (-r^{-1} v') x_2^{(i)} + (\tilde{r} r^{-2} r' v' - \tilde{r} r^{-1} v'') x_2^{(p-1)} \\
&= \sum_{i=0}^{p-1} k^i r^{-i} (-r^{-1} v') x_2^{(i)} + \tilde{r} (-r^{-1} v')' x_2^{(p-1)}.
\end{aligned}$$

This shows that $D_H(y) = D_H(-r^{-1} v')$, proving (2.3).

Setting $u = \tilde{r}$ in (2.2) now gives $[t_\mu, \varphi_k(v)] = k\varphi_k(v)$. Since $\varphi_k(v) \neq 0$ for all nonzero $v \in \mathcal{O}(2; (2, 1))[x_1]$, comparing dimensions yields that C_μ is spanned by h_μ , n_μ and C'_μ and that for every $k \in \mathbb{F}_p^\times$ the k -eigenspace of $\text{ad } t_\mu$ has dimension p^2 and is spanned by all $\varphi_k(v)$ with $v \in \mathcal{O}(2; (2, 1))[x_1]$.

(iii) Clearly, $n_\mu^{[p]} = D_1^{p^2} - \mu^p (x_2^{(p-1)} D_1)^p = 0$. Next observe that

$$[h_\mu, n_\mu] = [D_H(r^{-1} x_2 - x_2^{(p)}), D_1^p + \mu D_H(x_2^{(p)})] = \mu D_H((r^{-1})' x_2 \cdot x_2^{(p-1)}) = 0.$$

We claim that $h_\mu^{[p]} + \mu h_\mu + n_\mu = 0$. If $\mu = 0$, then $h_\mu = D_1 - x_2^{(p-1)} D_1$ and $n_\mu = D_1^p$; hence, our claim is true in this case. Assume now that $\mu \neq 0$ and set $q := h_\mu + \mu^{-1} n_\mu$. Since our remarks at the beginning of this part imply that $q^{[p]} = (h_\mu + \mu^{-1} n_\mu)^{[p]} = h_\mu^{[p]}$, we are reduced to showing that $q^{[p]} + \mu q = 0$. As $[D_H(x_1^{(p-1)} x_2), (\text{ad } D_1)^i (D_H(x_1^{(p-1)} x_2))] = 0$ for all $i \leq p-2$, we see that

$$\begin{aligned}
q^{[p]} &= (\mu^{-1} D_1^p - D_1 - \mu D_H(x_1^{(p-1)} x_2))^{[p]} = (-D_1 - \mu D_H(x_1^{(p-1)} x_2))^{[p]} \\
&= -D_1^p - (\text{ad } D_1)^{p-1} (\mu D_H(x_1^{(p-1)} x_2)) \\
&- \frac{1}{2} [\mu D_H(x_1^{(p-1)} x_2), (\text{ad } D_1)^{p-2} (\mu D_H(x_1^{(p-1)} x_2))] \\
&= -D_1^p + \mu D_1 + \mu^2 D_H(x_1^{(p-1)} x_2) = -\mu q,
\end{aligned}$$

and our claim follows.

(iv) Now suppose $\mu = 0$. Then $t_\mu = D_H(x_1(1+x_2^{(p-1)}))$, $h_\mu = D_H(x_2 - x_2^{(p)}) = (x_2^{(p-1)} - 1)D_1$ and $n_\mu = D_1^p$. Set $C := C_0$ and $C_{(0)} := C \cap G_{(0)}$. By Lemma 2.2(i), which we have already proved, C is spanned by D_1^p , $(x_2^{(p-1)} - 1)D_1$ and by all $D_H(x_1^{(k+1)} + x_1^{(k)} \tilde{r} x_2^{(p-1)})$ with $0 \leq k \leq p^2 - 1$. As a consequence, $C = F D_1^p \oplus F(x_2^{(p-1)} - 1)D_1 \oplus F t_\mu \oplus C_{(0)}$. As $G_{(0)}$ is a restricted subalgebra of \mathcal{G} , so is $C_{(0)}$. From this it is immediate that $C_{(0)}$ is a p -nilpotent subalgebra of \mathcal{G} . Note that $C \cap S = F t_\mu \oplus C_{(0)}$ is an ideal of C . Since $((x_2^{(p-1)} - 1)D_1)^{[p]} = -D_1^p$ and $(D_1^p)^{[p]} = 0$ (as derivations of S), Jacobson's formula

implies that $C^{[p]} \subset FD_1^p \oplus Ft_\mu \oplus C_{(0)}$ and $C^{[p]^2} \subset Ft_\mu \oplus C_{(0)}$. Since $C_{(0)}$ is p -nilpotent and $[t_\mu, C] = 0$, it follows that $C^{[p]^e} = Ft_\mu$ for all $e \gg 0$. Hence C is a restricted nilpotent subalgebra of \mathcal{G} and Ft_μ is the unique maximal torus of C . \square

If u belongs to the linear span of all $x_1^{(i)}$ with $2 \leq i \leq p^2$, then $r^{-1}u' \in \mathcal{O}(2; (2, 1))_{(1)}$, forcing $(r^{-1}u')^p = 0$. For $k \in \mathbb{F}_p^\times$ we write S_k for the k -eigenspace of $\text{ad } t_\mu$. In view of (2.2) we have that $(\text{ad } D_H(u + \tilde{r}u'x_2^{(p-1)}))^p(S_k) = (0)$ for all $k \in \mathbb{F}_p^\times$. Since

$$(\text{ad } D_H(u + \tilde{r}u'x_2^{(p-1)}))^p(C_\mu) \subset (\text{ad } D_H(u + \tilde{r}u'x_2^{(p-1)}))^{p-1}(C'_\mu) \subset (C'_\mu)^{(1)} = (0)$$

by Lemma 2.2(i), it follows that $(\text{ad } D_H(u + \tilde{r}u'x_2^{(p-1)}))^p = 0$. Therefore, for all u as above and $c \in F$ the exponential $\exp(c \text{ad } D_H(u + u'\tilde{r}x_2^{(p-1)}))$ is well-defined as a linear operator on S .

Lemma 2.3. *Suppose $\mu \neq 0$ and let $Z(t_\mu)$ denote the stabilizer of t_μ in $\text{Aut } S$.*

- (i) $\exp(c \text{ad } D_H(x_1^{(m)} + x_1^{(m-1)}\tilde{r}x_2^{(p-1)})) \in Z(t_\mu)$ for all $3 \leq m \leq p^2$.
- (ii) For every $h \in G \cap C_\mu$ with $h \notin C'_\mu$ there exist $z \in Z(t_\mu)$ and $a \in F^\times$ such that $z(h) = ah_\mu + bt_\mu + sD_H(x_1^{(p^2)})$ for some $b, s \in F$.
- (iii) If $h \in (G \cap C_\mu) \setminus C'_\mu$, then for every $k \in \mathbb{F}_p^\times$ there is $v_k \in 1 + \mathcal{O}(2; (2, 1))_{(1)}[x_1]$ such that $\varphi_k(v_k)$ is an eigenvector for $\text{ad } h$ and $\varphi_k(v_k)^{[p]}$ is a nonzero p -semisimple element of \mathcal{G} .
- (iv) For every $h \in (G \cap C_\mu) \setminus C'_\mu$ there exists a nonzero $x \in \mathfrak{c}_S(t_\mu)$ such that $\text{ad } x$ is not nilpotent and $[h, x] = \lambda x$ for some nonzero $\lambda \in F$.

Proof. (a) For $1 \leq m \leq p^2$ set $\mathcal{D}_m := \text{ad } D_H(x_1^{(m)} + x_1^{(m-1)}\tilde{r}x_2^{(p-1)})$. As $(\text{ad } \mathcal{D}_m)^p = 0$ for $m \geq 3$, in order to prove (i) it suffices to show that

$$(2.4) \quad \sum_{i=1}^{p-1} \frac{1}{i!(p-i)!} [\mathcal{D}_m^i(y_1), \mathcal{D}_m^{p-i}(y_2)] = 0 \quad (\forall y_1, y_2 \in S, \forall m \geq 3).$$

It follows from Lemma 2.2(i) that $\mathcal{D}_m^2(C_\mu) \subset (C'_\mu)^{(1)} = (0)$. Therefore, we just need to show that (2.4) holds for all $y_1 = \varphi_k(v_1)$ and $y_2 = \varphi_l(v_2)$, where $k, l \in \mathbb{F}_p^\times$ and $v_1, v_2 \in \mathcal{O}(2; (2, 1))[x_1]$.

For $3 \leq m \leq p$ we have $(r^{-1}x_1^{(m-1)})^{(p+1)/2} = 0$, since $\mathcal{O}(2; (1, 1))$ is a subalgebra of $\mathcal{O}(2; (2, 1))$ and $\frac{(m-1)(p+1)}{2} > p$. In light of (2.2) this gives $(\text{ad } \mathcal{D}_m)^{(p+1)/2}(\varphi_i(v)) = 0$ for all $i \in \mathbb{F}_p^\times$ and $v \in \mathcal{O}(2; (2, 1))[x_1]$. Hence (2.4) holds for $m \leq p$.

If $m \geq p+2$, then (2.2) yields that $\mathcal{D}_m^i(\varphi_k(v_1)) = \varphi_k(w_1)$ and $\mathcal{D}_m^{p-i}(\varphi_l(v_2)) = \varphi_l(w_2)$ for some $w_1 \in \mathcal{O}(2; (2, 1))_{((p+1))}[x_1]$ and $w_2 \in \mathcal{O}(2; (2, 1))_{((p-i)(p+1))}[x_1]$. As $[\varphi_k(w_1), \varphi_l(w_2)] = 0$ in this case, we deduce that (2.4) holds for $m \geq p+2$. As $\mathcal{O}(2; (2, 1))_{(p^2)}[x_1] = 0$, this argument also shows that (2.4) holds if $m = p+1$ and either v_1 or v_2 belongs to $\mathcal{O}(2; (2, 1))_{(1)}[x_1]$.

Thus, in order to prove (i) it suffices to show that (2.4) holds for $m = p+1$ and $v_1 = v_2 = 1$. Suppose the contrary and set

$$Y := \sum_{i=1}^{p-1} \frac{1}{i!(p-i)!} [\mathcal{D}_{p+1}^i(\varphi_k(1)), \mathcal{D}_{p+1}^{p-i}(\varphi_l(1))].$$

Arguing as in the preceding paragraph we now observe that Y is a nonzero multiple of either $\varphi_{k+l}(x_1^{(p^2-1)})$ (if $k+l \neq 0$) or $D_H(x_1^{(p^2-1)}(1-x_2^{(p-1)}))$ (if $k+l=0$). In any event, $(\text{ad } n_\mu)^{p-1}(Y) \neq 0$.

Set $N_\mu := \text{ad } n_\mu$. We know from the proof of Lemma 2.2 that $[N_\mu, \mathcal{D}_{p+1}] = \mathcal{D}_1$, $[\mathcal{D}_1, \mathcal{D}_{p+1}] = 0$ and $N_\mu(\varphi_i(1)) = 0$ for all $i \in \mathbb{F}_p^\times$. From this it follows that

$$\begin{aligned}
N_\mu^{p-1}(Y) &= \sum_{i=1}^{p-1} \frac{1}{i!(p-i)!} \left(\sum_{j=0}^{p-1} (-1)^j \left[N_\mu^j(\mathcal{D}_{p+1}^i(\varphi_k(1))), N_\mu^{p-1-j}(\mathcal{D}_{p+1}^{p-i}(\varphi_l(1))) \right] \right) \\
&= \sum_{i=1}^{p-1} (-1)^i [\mathcal{D}_1^i(\varphi_k(1)), (\mathcal{D}_1^{p-i-1} \mathcal{D}_{p+1})(\varphi_l(1))] \\
&\quad + \sum_{i=1}^{p-1} (-1)^{i-1} [(\mathcal{D}_1^{i-1} \mathcal{D}_{p+1})(\varphi_k(1)), \mathcal{D}_1^{p-i}(\varphi_l(1))] \\
&= \mathcal{D}_1^{p-1}([\varphi_k(1), \mathcal{D}_{p+1}(\varphi_l(1))]) - [\varphi_k(1), (\mathcal{D}_1^{p-1} \mathcal{D}_{p+1})(\varphi_l(1))] \\
&\quad + \mathcal{D}_1^{p-1}([\mathcal{D}_{p+1}(\varphi_k(1)), (\varphi_l(1))]) - [(\mathcal{D}_1^{p-1} \mathcal{D}_{p+1})(\varphi_k(1)), \varphi_l(1)] \\
&= (\mathcal{D}_1^{p-1} \mathcal{D}_{p+1})([\varphi_k(1), \varphi_l(1)]) - l[\varphi_k(1), \varphi_l(x_1^{(p)})] - k[\varphi_k(x_1^{(p)}), \varphi_l(1)]
\end{aligned}$$

(we used (2.2) and the equalities $r^p = 1$, $k^p = k$ and $l^p = l$). On the other hand, comparing components of x_2 -degree 0 and 1 one observes that

$$[\varphi_k(u), \varphi_l(v)] = \begin{cases} \varphi_{k+l}((lu'v - kuv')r^{-1}) & \text{if } k+l \neq 0, \\ kD_H((u'v - uv') + (u'v - uv')'\tilde{r}x_2^{p-1}) & \text{if } k+l = 0 \end{cases}$$

for all $u, v \in \mathcal{O}(2; (2, 1))[x_1]$. But then $l[\varphi_k(1), \varphi_l(x_1^{(p)})] + k[\varphi_k(x_1^{(p)}), \varphi_l(1)] = 0$ and $[\varphi_k(1), \varphi_l(1)] = 0$, forcing $N_\mu^{p-1}(Y) = 0$, a contradiction. Statement (i) follows.

(b) Observe that $C_\mu \cap G = C'_\mu \oplus Fh_\mu$. If $h \in C_\mu \cap G$ and $h \notin C'_\mu$, then Lemma 2.2(i) implies that there are $a \in F^\times$, $b, s \in F$ such that $h = ah_\mu + bt_\mu + sD_H(x_1^{(p^2)}) + \sum_{i=2}^{p^2-1} a_i D_H(x_1^{(i)} + x_1^{(i-1)}\tilde{r}x_2^{(p-1)})$ for some $a_i \in F$. Since C'_μ is abelian, r is invertible, and

$$(\exp a_i \mathcal{D}_m)(h_\mu) = h_\mu + a_i D_H(r^{-1}(x_1^{(m-1)} + x_1^{(m-2)}\tilde{r}x_2^{(p-1)})) \quad (3 \leq m \leq p^2)$$

by (2.1), we can clear the a_i 's by applying suitable automorphisms from $Z(t_\mu)$. This proves statement (ii).

In dealing with (iii) we may assume that $h = h_\mu + sD_H(x_1^{(p^2)})$ where $s \in F$. In view of (2.3) we need to find $v_k = 1 + b_1x_1^{(1)} + b_2x_1^{(2)} + \cdots + b_{p^2-1}x_1^{(p^2-1)}$ and $\eta_k \in F$ satisfying the condition

$$\begin{aligned}
\eta_k \varphi_k(v_k) &= [h_\mu + sD_H(x_1^{(p^2)}), \varphi_k(v_k)] \\
&= -\varphi_k(r^{-1}v'_k) + sD_H\left(x_1^{(p^2-1)} \cdot \left(\sum_{i=0}^{p-1} k^i r^{-i} v_k x_2^{(i-1)} + k^{p-1} \tilde{r} v'_k x_2^{(p-2)}\right)\right) \\
&= -\varphi_k(r^{-1}v'_k) + sk\varphi_k(x_1^{(p^2-1)}v_k).
\end{aligned}$$

This holds if and only if

$$-b_1 - b_2x_1 - \cdots - b_{p^2-1}x_1^{(p^2-2)} + skx_1^{(p^2-1)} = \eta_k r(1 + b_1x_1^{(1)} + \cdots + b_{p^2-1}x_1^{(p^2-1)}).$$

Because

$$\begin{aligned} r(1 + \sum_{i=1}^{p^2-1} b_i x_1^{(i)}) &= (1 + \sum_{i=1}^{p^2-1} b_i x_1^{(i)}) + \mu(x_1^{(p-1)} + \sum_{i=1}^{p-1} b_{ip} x_1^{(ip+p-1)}) \\ &= (1 + \sum_{i=1}^{p^2-1} b_i x_1^{(i)}) + \mu \sum_{i=0}^{p-1} b_{ip} x_1^{(ip+p-1)} \end{aligned}$$

by Lucas' theorem, this leads to the system of equations

$$\begin{aligned} b_0 &= 1; \\ b_i &= -\eta_k b_{i-1} & 1 \leq i \leq p^2 - 1, \quad i \notin p\mathbb{Z}; \\ b_{ip} &= -\eta_k(b_{ip-1} + \mu b_{i-1}) & 1 \leq i \leq p-1; \\ \eta_k b_{p^2-1} &= sk. \end{aligned}$$

Arguing recursively, one observes that there is a bijection between the solutions to this system and the roots of a polynomial of the form $X^{p^2} + \sum_{i=1}^{p^2-1} \lambda_i X^i - sk$, where $\lambda_i \in F$. Since F is algebraically closed, it follows that our eigenvalue problem has at least one solution.

(c) In view of our discussion in part (b), $\varphi_k(v_k) \equiv D_H(x_2) + b_1 D_H(x_1) \pmod{S_{(0)}}$. Since $D_H(x_2) = -D_1$ and $S_{(0)}$ is a restricted subalgebra of \mathfrak{G} , Jacobson's formula shows that $\varphi_k(v_k)^{[p]} = -D_1^p + w_k$ for some $w_k \in S$. In particular, $\varphi_k(v_k)^{[p]} \neq 0$. Note that $\varphi_k(v_k)^{[p]} \in C'_\mu \cap S_p \cap \ker \text{ad } h$. Now, using (2.1) it is easy to observe that $C'_\mu \cap \ker \text{ad } h = Ft_\mu$, whilst from Lemma 2.2 it is immediate that $C_\mu \cap S_p = F(\mu h_\mu + n_\mu)$. Lemma 2.2 also implies that $\mu h_\mu + n_\mu = -h_\mu^{[p]}$ and $h_\mu^{[p]^2} = -\mu^p h_\mu^{[p]}$.

Let h_s denote the p -semisimple part of h in \mathfrak{G} , an element of $C'_\mu \cap \ker \text{ad } h \cap S_p$. Since the above discussion shows that $C'_\mu \cap S_p \cap \ker \text{ad } h$ has dimension ≤ 2 , in order to finish the proof of (iii) we need to show that t_μ and h_s are linearly independent.

Suppose the contrary. Then $\text{ad } h$ acts nilpotently on C'_μ . Recall that $h \in h_\mu + C'_\mu$ and C'_μ is abelian. So $\text{ad } h_\mu$ acts on C'_μ nilpotently, too. Since $\mu \neq 0$, our earlier remarks and Lemma 2.2(iii) now show that $\text{ad}(h_\mu^{[p]}) = -\mu \text{ad } h_\mu - \text{ad } n_\mu$ acts trivially on C'_μ . Since this violates (2.1), we reach a contradiction. Statement (iii) follows.

(d) In proving (iv) we may assume that $h = h_\mu + sD_H(x_1^{(p^2)})$; see part (b). We claim that there exist $u = x_1 + c_1 x_1^{(2)} + \cdots + c_{p^2-2} x_1^{(p^2-1)}$ and $\lambda \in F^\times$ such that

$$[h, D_H(u + u' \tilde{r} x_2^{(p-1)})] = \lambda D_H(u + u' \tilde{r} x_2^{(p-1)}).$$

Since C'_μ is abelian, it follows from (2.1) that

$$[h, D_H(u + u' \tilde{r} x_2^{(p-1)})] = [h_\mu, D_H(u + u' \tilde{r} x_2^{(p-1)})] = -D_H(r^{-1} u' + (r^{-1} u')' \tilde{r} x_2^{(p-1)}).$$

Thus, we seek u such that $r^{-1} u' = a - \lambda u$ for some $a \in F$. Since $r^{-1} = 1 - \mu x_1^{(p-1)}$, this entails that $a = 1$, $c_1 = -\lambda$, and

$$(2.5) \quad (1 - \mu x_1^{(p-1)}) \left(1 + \sum_{i=1}^{p^2-2} c_i x_1^{(i)} \right) = 1 + c_1 \left(x_1 + \sum_{i=1}^{p^2-2} c_i x_1^{(i+1)} \right).$$

Since $x_1^{(p-1)} \cdot (1 + \sum_{i=1}^{p^2-2} c_i x_1^{(i)}) = (x_1^{(p-1)} + \sum_{i=1}^{p-1} c_{ip} x_1^{(ip+p-1)})$ by Lucas' theorem, we see that $c_{i+1} = c_1 c_i$ if $p \nmid (i+2)$. Induction on k shows that $c_{kp+p-1} = c_1^k (c_1^{p-1} + \mu)^{k+1}$ for $0 \leq k \leq p-1$. As $c_{p^2-1} = 0$, this yields $c_1^{p-1} (c_1^{p-1} + \mu)^p = 0$. As $c_1 = -\lambda \neq 0$, we see that c_1 must satisfy the equation $X^{p-1} + \mu = 0$. Conversely, any root of this

equation gives rise to a solution of (2.5) with $\lambda = -c_1 \neq 0$ (recall that $\mu \neq 0$ by our assumption). The claim follows.

We now set $x := D_H(u + u'\tilde{r}x_2^{(p-1)})$, where u is as above. Clearly, $x \in S$. Since $r^{-1}u' - 1 \in \mathcal{O}_{(1)}(2; (2, 1))$, it follows from (2.2) that $(\text{ad } x)^p(\varphi_k(v)) = k^p \varphi((r^{-1}u')^p v) = k \varphi_k(v)$ for all $v \in \mathcal{O}(2; (2, 1))[x_1]$ and all $k \in \mathbb{F}_p$. This implies that $\text{ad } x$ is not nilpotent, completing the proof. \square

We now let \mathfrak{t} be a 2-dimensional torus in \mathcal{G} .

Lemma 2.4. *There exist nonzero $u_1, u_2 \in S$ such that $\mathfrak{t} = F(D_1^p + u_1) \oplus F u_2$.*

Proof. Since $V^{[p]} = 0$, the restricted Lie algebra \mathcal{G}/S_p is p -nilpotent. As \mathfrak{t} is a torus, it must be that $\mathfrak{t} \subset S_p$. Then $\mathfrak{t} \cap S \neq (0)$, for $\dim \mathfrak{t} = 2$.

Suppose $\mathfrak{t} \subset S$. Since $S_{(0)}/S_{(1)} \cong \mathfrak{sl}(2)$ and $S_{(-1)}/S_{(0)}$ is a 2-dimensional irreducible module over $S_{(0)}/S_{(1)}$, every nonzero element of $\mathfrak{t} \cap S_{(0)}$ acts invertibly on $S_{(-1)}/S_{(0)}$. Therefore, $\mathfrak{t} \cap S_{(0)} \neq (0)$ would force $\mathfrak{t} \subset S_{(0)}$, which is false because $S_{(0)}$ has toral rank 1 in S . On the other hand, if $\mathfrak{t} \cap S_{(0)} = (0)$ (and still $\mathfrak{t} \subset S$), then \mathfrak{t} would contain an element of the form $D_1 + u$ with $u \in S_{(0)}$. But this would yield $D_1^p \in \mathfrak{t} + S = S$, as $S_{(0)}$ is a restricted subalgebra of S_p . Therefore, $\mathfrak{t} \not\subset S$. Since D_1 is nilpotent and S has codimension 1 in S_p , our statement follows immediately. \square

Lemma 2.5. *Let $\mathfrak{h} = \mathfrak{c}_S(\mathfrak{t})$ and let $\alpha \in \Gamma(S, \mathfrak{t})$.*

- (1) *If α vanishes on \mathfrak{h} , then $G(\alpha)$ is solvable.*
- (2) *If α does not vanish on \mathfrak{h} , then $G(\alpha) \cong H(2; \underline{1})$.*
- (3) *$\dim G_\gamma = p + \delta_{\gamma, 0}$ for all $\gamma \in \Gamma(G, \mathfrak{t}) \cup \{0\}$.*
- (4) *$\Gamma(S, \mathfrak{t}) \cup \{0\}$ is a two-dimensional vector space over \mathbb{F}_p .*

Proof. Note that $\mathfrak{c}_{S_p}(\mathfrak{t}) = \mathfrak{t} + \mathfrak{h}$ and \mathfrak{t} is a standard torus of maximal dimension in S_p . Therefore, the results of [B-W 88, (10.1.1)] and [St 91, (VI)] apply to \mathfrak{t} .

If α does not vanish on \mathfrak{h} , then $G(\alpha) \cong H(2; \underline{1})$ by [P-St 04, Prop. 2.1(2)]. Suppose $\alpha(\mathfrak{h}) = 0$. As \mathfrak{t} is a maximal torus of S_p , we have that $\alpha(L_{i\alpha}^{[p]}) = 0$ for all $i \in \mathbb{F}_p^\times$. Then $S(\alpha)$ is nilpotent due to the Engel–Jacobson theorem. As G/S is nilpotent too, we conclude that $G(\alpha)$ is solvable.

By [B-W 88, (10.1.1(e))], there is a 2-dimensional torus \mathfrak{t}' in S_p such that all roots in $\Gamma(S, \mathfrak{t}')$ are proper. Then [St 91, (VI.2(2))] applies showing that all root spaces of G with respect to \mathfrak{t}' are p -dimensional and $\dim \mathfrak{c}_G(\mathfrak{t}') = p + 1$. By [P 89], all root spaces of G with respect to \mathfrak{t} must have the same property, and $\dim \mathfrak{c}_G(\mathfrak{t}) = p + 1$ (see also [P-St 99, Cor. 2.11]). As $\dim S = p^3 - 2$ and $\dim S_\gamma \leq p$ for all $\gamma \in \Gamma(S, \mathfrak{t})$, we derive that $|\Gamma(S, \mathfrak{t})| = p^2 - 1$. As a consequence, the set $\Gamma(S, \mathfrak{t}) \cup \{0\}$ is 2-dimensional vector space over \mathbb{F}_p . This completes the proof. \square

Lemma 2.6. *Under the above assumptions on \mathfrak{t} and S the following hold:*

- (1) *If $TR(\mathfrak{h}, S) = 2$, then all roots in $\Gamma(S, \mathfrak{t})$ are Hamiltonian improper.*
- (2) *If $TR(\mathfrak{h}, S) = 1$, then $\Gamma(S, \mathfrak{t})$ contains a solvable root.*
- (3) *Suppose that $TR(\mathfrak{h}, S) = 1$ and $\mathfrak{h}_p \cap S_{(0)}$ contains a nonnilpotent element. Then for any solvable $\alpha \in \Gamma(S, \mathfrak{t})$ the 1-section $G(\alpha)$ is nilpotent.*

Proof. Suppose $TR(\mathfrak{h}, S) = 2$. Then no root in $\Gamma(S, \mathfrak{t})$ vanishes on \mathfrak{h} ; hence, all roots in $\Gamma(S, \mathfrak{t})$ are Hamiltonian by Proposition 2.5(2). If $\mathfrak{h} \cap S_{(0)}$ contains a nonnilpotent element, x say, then the image of x in $S_{(0)}/S_{(1)} \cong \mathfrak{sl}(2)$ acts invertibly on $S_{(-1)}/S_{(0)}$. As \mathfrak{h} is nilpotent, this would force $\mathfrak{h} \subset S_{(0)}$, and hence $TR(\mathfrak{h}, S) = 1$, a contradiction. Consequently, $\mathfrak{t} \cap S_{(0)} = (0)$. By [B-W 88, (10.1.1(d))] (see the proof on p. 232-233), every Hamiltonian root is then improper.

Now suppose $TR(\mathfrak{h}, S) = 1$. Then the unique maximal torus of \mathfrak{h}_p is spanned by a toral element, hence it follows from Lemma 2.5(4) that there is a root in $\Gamma(S, \mathfrak{t})$ which vanishes on \mathfrak{h} . Every such root is solvable by Proposition 2.5(1).

Finally, suppose that $TR(\mathfrak{h}, S) = 1$ and $\mathfrak{h}_p \cap S_{(0)}$ contains a nonnilpotent element. Since $S_{(0)}$ is a restricted subalgebra of S_p , we then have $S_{(0)} \cap \mathfrak{h}_p \cap \mathfrak{t} \neq (0)$. Since $\mathfrak{t} \cap S = Fu_2$ for some nonzero $u_2 \in S$ (see Lemma 2.4), it must be that $u_2 \in S_{(0)}$ and $u_2^{[p]} \in Fu_2$.

If $\alpha \in \Gamma(S, \mathfrak{t})$ is solvable, then $\alpha(\mathfrak{h}) = 0$ by Lemma 2.5(2). As explained in the proof of Lemma 2.5 the Lie algebra $S(\alpha)$ is nilpotent. There exists an element $t \in F^\times u_2$ with $t^{[p]} = t$ such that $G(\alpha) = \mathfrak{c}_G(t)$. Set $W := \{v - (\text{ad } t)^{p-1}(v) \mid v \in V\}$. By construction, $W \subset \mathfrak{c}_G(t)$ and $G = W \oplus S$. Since $V \subset G_{(1)}$ and $t \in S_{(0)}$, we have the inclusion $W \subset G_{(1)}$. In particular, all elements of W act nilpotently on $\mathfrak{c}_G(t)$.

Since $S(\alpha)$ is a nilpotent ideal of $G(\alpha)$, the set $(\text{ad}_{G(\alpha)} S(\alpha)) \cup (\text{ad}_{G(\alpha)} W)$ is weakly closed and consists of nilpotent endomorphisms. Since $G(\alpha) = W \oplus S(\alpha)$, the Engel–Jacobson theorem now shows that $G(\alpha)$ is nilpotent. \square

Lemma 2.7. *If $t \in S_p$ is a toral element not contained in S , then t is conjugate to $D_1^p + D_1 + D_H(x_1 x_2)$ under the automorphism group of S .*

Proof. By our assumption, $t = aD_1^p + w$ for some $a \in F^\times$ and $w \in S$. Choose $\alpha \in F$ satisfying $\alpha^p = a$ and let σ_α denote the automorphism of S which sends $D_H(x_1^{(i)} x_2^{(j)})$ to $\alpha^{i-1} D_H(x_1^{(i)} x_2^{(j)})$; see [St 04, Thm. 7.3.6]. Then $\sigma_\alpha(t) = -aD_H(\alpha^{-1} x_2)^p + w'$ for some $w' \in S$. Hence we may assume that $a = 1$. The description of $\text{Aut } S$ given in [St 04, Thms 7.3.5 & 7.3.2] shows that for any pair of nonnegative integers $(m, n) \neq (p, 1)$ such that either $p \leq m < p^2$ and $n < p$ or $(m, n) = (p^2, 0)$ and any $\lambda \in F$ there is $\sigma_{m,n,\lambda} \in \text{Aut } S$ such that $\sigma_{m,n,\lambda}(u) \equiv u + \lambda[D_H(x_1^{(m)} x_2^{(n)}), u] \pmod{S_{i+(m+n-1)}}$ for all $u \in S_{(i)}$. Using Jacobson’s formula (with $u = D_1$) it is not hard to observe that

$$\sigma_{m,n,\lambda}(D_1^p) \equiv D_1^p - \lambda D_H(x_1^{(m-p)} x_2^{(n)}) \pmod{S_{(m+n-p-1)}}.$$

This implies that there exists $g \in \text{Aut } S$ such that $g(t) = D_1^p + bD_1 + D_H(x_1^{(p^2-p)} \psi)$ for some $\psi \in F[x_1, x_2] \subset \mathcal{O}(2; (1, 1))$ with $\psi(0) = 0$. Write $\psi = \sum_{i=0}^{p-1} \psi_i x_1^{(i)}$ with $\psi_i \in F[x_2]$, where $\psi_0(0) = 0$. The element $g(t)$ being toral, it must be that $b = 1$. Note that $(\text{ad } D_H(x_1^{(p^2-p)} \psi))(\text{ad } (D_1^p + D_1))^i (D_H(x_1^{(p^2-p)} \psi)) = 0$ for $0 \leq i \leq p-3$ and

$$(\text{ad } D_H(x_1^{(p^2-p)} \psi))(\text{ad } (D_1^p + D_1))^{p-2} (D_H(x_1^{(p^2-p)} \psi)) = [D_H(x_1^{(p^2-p)} \psi), D_H(x_1^{(p)} \psi)].$$

Because

$$(\text{ad } D_1^p + \text{ad } D_1)^{p-1} = \sum_{i=0}^{p-1} (-1)^i (\text{ad } D_1)^{pi} (\text{ad } D_1)^{p-i-1} = \sum_{i=1}^p (-1)^{i-1} (\text{ad } D_1)^{i(p-1)}$$

and $D_1^p(\psi) = 0$, Jacobson's formula yields that

$$\begin{aligned} g(t)^{[p]} &= (D_1^p + D_1)^{[p]} + (\text{ad}(D_1^p + D_1))^{p-1}(D_H(x_1^{(p^2-p)}\psi)) \\ &\quad + \frac{1}{2}[D_H(x_1^{(p^2-p)}\psi), D_H(x_1^{(p)}\psi)] \\ &= D_1^p + D_H(\psi) - D_H(x_1^{(p-1)}\psi) + \sum_{i \geq p} D_H(x_1^{(i)}q_i) \end{aligned}$$

for some $q_i \in F[x_2]$. As the RHS equals $D_1^p - D_H(x_2) + D_H(x_1^{(p^2-p)}\psi)$ and $x_1^{(p-1)}\psi = x_1^{(p-1)}\psi_0$, we derive that $\psi_0 = -x_2$, $\psi_i = 0$ for $1 \leq i \leq p-2$, and $\psi_{p-1} = \psi_0$. In other words, $\psi = -(1 + x_1^{(p-1)})x_2$ and

$$g(t) = (D_1^p + D_1) - D_H(x_1^{(p^2-p)}x_2) - D_H(x_1^{(p^2-1)}x_2).$$

Next we show that this element is toral. Note that

$$\begin{aligned} (D_1^p + D_1) - D_H(x_1^{(p^2-p)}x_2) - D_H(x_1^{(p^2-1)}x_2) &= (D_1^p + D_1) - [D_1^p + D_1, D_H(x_1^{(p^2)}x_2)] \\ \text{and } \binom{p^2-1}{p} - \binom{p^2-1}{p-1} &= \binom{p-1}{1} - 1 = -2 \text{ by Lucas' theorem. Then} \\ [D_H(x_1^{(p^2-p)}(1 + x_1^{(p-1)})x_2), D_H(x_1^{(p)}(1 + x_1^{(p-1)})x_2)] &= [D_H(x_1^{(p^2-p)}x_2), D_H(x_1^{(p)}x_2)] \\ &= -2D_H(x_1^{(p^2-1)}x_2). \end{aligned}$$

In view of the earlier computations this gives

$$\begin{aligned} &((D_1^p + D_1 - D_H(x_1^{(p^2-p)}x_2) - D_H(x_1^{(p^2-1)}x_2)))^{[p]} \\ &= D_1^p - (\text{ad}(D_1^p + D_1))^p(D_H(x_1^{(p^2)}x_2)) - D_H(x_1^{(p^2-p)}x_2) \\ &= D_1^p - D_H(x_2) - D_H(x_1^{(p^2-p)}x_2) - D_H(x_1^{(p^2-1)}x_2). \end{aligned}$$

So the element $D_1^p + D_1 - D_H((x_1^{(p^2-p)} + x_1^{(p^2-1)})x_2)$ is indeed toral.

As a result, all toral elements in $S_p \setminus S$ are conjugate under $\text{Aut } S$. To finish the proof it remains to note that the element $D_1^p + D_1 + D_H(x_1x_2) \in S_p \setminus S$ is toral. \square

3. Two-sections in simple Lie algebras

In this section our standing hypothesis is that L is a finite-dimensional simple Lie algebra and T is a torus of maximal dimension in the semisimple p -envelope L_p of L . Given $\alpha_1, \dots, \alpha_s \in \Gamma(L, T)$ we denote by $\text{rad}_T L(\alpha_1, \dots, \alpha_s)$ the maximal T -invariant solvable ideal of the s -section $L(\alpha_1, \dots, \alpha_s)$ and put

$$(3.1) \quad L[\alpha_1, \dots, \alpha_s] := L(\alpha_1, \dots, \alpha_s) / \text{rad}_T L(\alpha_1, \dots, \alpha_s).$$

We let $\tilde{S} = \tilde{S}(\alpha_1, \dots, \alpha_s)$ be the T -socle of $L[\alpha_1, \dots, \alpha_s]$, the sum of all minimal T -stable ideals of the Lie algebra $L[\alpha_1, \dots, \alpha_s]$. Then $\tilde{S} = \bigoplus_{i=1}^r \tilde{S}_i$ where each \tilde{S}_i is a *minimal* T -stable ideal of $L[\alpha_1, \dots, \alpha_s]$. It is immediate from the definition that both T and $L(\alpha_1, \dots, \alpha_s)_p$ act on $L[\alpha_1, \dots, \alpha_s]$ as derivations and preserve \tilde{S} . Thus, there is a natural restricted Lie algebra homomorphism $T + L(\alpha_1, \dots, \alpha_s)_p \rightarrow \text{Der } \tilde{S}$ which will be denoted by $\Psi_{\alpha_1, \dots, \alpha_s}$. Note that $L(\alpha_1, \dots, \alpha_s) \cap \ker \Psi_{\alpha_1, \dots, \alpha_s} = \text{rad}_T L(\alpha_1, \dots, \alpha_s)$ and, moreover, the image of $\Psi_{\alpha_1, \dots, \alpha_s}$ can be identified with a semisimple restricted Lie subalgebra of $\text{Der } \tilde{S}$ containing $L[\alpha_1, \dots, \alpha_s]$ as an ideal.

We often regard the linear functions on T as functions on the nilpotent restricted Lie algebra $\mathfrak{c}_{L_p}(T)$ by using the rule $\gamma(x) := (\gamma(x^{[p]^e}))^{p^{-e}}$ for all $x \in \mathfrak{c}_{L_p}(T)$, where $e \gg 0$ (this makes sense because T coincides with the set of all p -semisimple elements of $\mathfrak{c}_{L_p}(T)$).

Let $\text{nil } H_p$ denote the maximal p -nilpotent ideal of the restricted Lie algebra H_p . According to [P-St 04, Cor. 3.9], the inclusion $H^4 \subset \text{nil } H_p$ holds and all roots in $\Gamma(L, T)$ are linear functions on H .

Lemma 3.1. *If $\delta \in \Gamma(L, T)$ has the property that $\delta(H) \neq 0$, then $\delta([L_\delta, L_{-\delta}]^2) = 0$ and $[L_\delta, L_{-\delta}]^3 \subset \text{nil } H_p$.*

Proof. This is immediate from [P-St 04, Prop. 3.4]. \square

Proposition 3.2. *Let \mathfrak{t} be a torus in L_p whose centralizer in L is nilpotent, and assume further that \mathfrak{t} contains the all p -semisimple elements of the p -envelope of $\mathfrak{c}_L(\mathfrak{t})$ in L_p . Let $\eta \in \Gamma(L, \mathfrak{t})$ be such that $L(\eta)$ is nonsolvable and denote by $S(\eta)$ the socle of the semisimple Lie algebra $L(\eta)/\text{rad } L(\eta)$. Then the following hold:*

- (1) *the radical $\text{rad } L(\eta)$ is \mathfrak{t} -stable;*
- (2) *the socle $S(\eta)$ is a simple Lie algebra invariant under the action of \mathfrak{t} ;*
- (3) *the centralizer $\mathfrak{c}_S(\mathfrak{t})$ is a Cartan subalgebra of toral rank 1 in S .*

Proof. The torus \mathfrak{t} satisfies the conditions of [P-St 04, Thm. 3.6]. Moreover, our first statement is nothing but [P-St 04, Thm. 3.6(1)]. The last two statements are immediate consequences of [P-St 04, Thm. 3.6(3)] and [P-St 04, Thm. 3.6(4)]. \square

Theorem 3.3. *For every $\gamma \in \Gamma(L, T)$ the radical $\text{rad } L(\gamma)$ is T -stable and either $L[\gamma]$ is one of (0) , $\mathfrak{sl}(2)$, $W(1; \underline{1})$, $H(2; \underline{1})^{(2)}$, $H(2; \underline{1})^{(1)}$ or $p = 5$, L_p possesses nonstandard tori of maximal dimension, and $L[\gamma] \cong H(2; \underline{1})^{(2)} \oplus F(1+x_1)^4 \partial_2$. If γ is nonsolvable, then the derived subalgebra $L[\gamma]^{(1)}$ is simple.*

Proof. This is immediate from [P-St 04, Cor. 3.7]. \square

Lemma 3.4. *Let $\mathfrak{g} = H(2; \underline{1})^{(2)} \oplus F(1+x_1)^{p-1} \partial_2$ and \mathfrak{h} a Cartan subalgebra of \mathfrak{g} . Then either \mathfrak{h} is abelian or \mathfrak{h}^3 contains a nonzero toral element of \mathfrak{g} .*

Proof. We regard \mathfrak{g} as a restricted Lie subalgebra of $\tilde{\mathfrak{g}} := H(2; \underline{1})$. Recall that $\tilde{\mathfrak{g}} = H(2; \underline{1})^{(2)} \oplus FD_H(x_1^{(p)}) \oplus FD_H(x_2^{(p)}) \oplus FD_H(x_1^{(p-1)} x_2^{(p-1)})$. Since $\tilde{\mathfrak{g}}^{[p]} \subset H(2; \underline{1})^{(2)}$ by Jacobson's formula, \mathfrak{h} coincides with $\mathfrak{c}_{\tilde{\mathfrak{g}}}(y)$ for some nonzero toral element $y \in H(2; \underline{1})^{(2)}$. By a result of Demuškin, there is $\sigma \in \text{Aut } H(2; \underline{1})^{(2)}$ such that either $\sigma(y) = D_H((1+x_1)x_2)$ or $\sigma(y)$ is a nonzero multiple of $D_H(x_1 x_2)$; see [St 04, Thm. 7.5.8]. In the latter case, there exist $a, b \in F$ such that $\sigma(\mathfrak{h})$ is contained in the span of $aD_H(x_1^{(p)}) + bD_H(x_2^{(p)})$ and all $D_H(x_1^{(i)} x_2^{(i)})$ with $1 \leq i \leq p-1$, hence is abelian. Then \mathfrak{h} is abelian, too. So assume we are in the former case. Then there are $a, b, c \in F$ such that $\sigma(\mathfrak{h})$ coincides with the span of all $D_H((1+x_1)^i x_2^{(i)})$ with $1 \leq i \leq p-2$ and $z := a(1+x_1)^{p-1} D_2 + bD_H(x_2^{(p)}) + cD_H((1+x_1)^{p-1} x_2^{(p-1)})$. If $a = 0$, then it is easy to check that $\sigma(\mathfrak{h})$ is abelian, whilst if $a \neq 0$, then $(\text{ad } z)^2(D_H((1+x_1)^3 x_2^{(3)}))$ is a nonzero multiple of $\sigma(y)$. This completes the proof. \square

Next we recall our results on 2-sections of L with respect to T . Let $\alpha, \beta \in \Gamma(L, T)$ be such that $L(\alpha, \beta)$ is nonsolvable. As explained in [P-St 04, p. 793], the T -socle

$\tilde{S} = \tilde{S}(\alpha, \beta)$ is either a unique minimal ideal of $L[\alpha, \beta]$ or $\tilde{S} = \tilde{S}_1 \oplus \tilde{S}_2$, where $TR(\tilde{S}_i) = 1$ for $i = 1, 2$ and each \tilde{S}_i is T -stable. Moreover, in the latter case the following holds:

Theorem 3.5 (cf. [P-St 04, Thm. 4.1]). *If $\tilde{S} = \tilde{S}_1 \oplus \tilde{S}_2$, then there exist $\delta_1, \delta_2 \in \Gamma(L, T)$ such that*

$$L[\delta_1]^{(1)} \oplus L[\delta_2]^{(1)} \subset L[\alpha, \beta] \subset L[\delta_1] \oplus L[\delta_2].$$

When the T -socle \tilde{S} is a minimal ideal of $L[\alpha, \beta]$, we have two possibilities: either $TR(\tilde{S}) = 2$ or $TR(\tilde{S}) = 1$.

Theorem 3.6. *Suppose \tilde{S} is the unique minimal ideal of $L(\alpha, \beta)$ and $TR(\tilde{S}) = 2$. Then \tilde{S} is simple, $\Psi_{\alpha, \beta}(L_\gamma) \subset \tilde{S}$ for all $\gamma \in \Gamma(L, T)$, and one of the following holds:*

- (1) \tilde{S} is one of $W(2; \underline{1})$, $S(3; \underline{1})^{(1)}$, $H(4; \underline{1})^{(1)}$, $K(3; \underline{1})^{(1)}$ and $L[\alpha, \beta] = \tilde{S}$;
- (2) \tilde{S} is one of $W(1; \underline{2})$, $H(2; \underline{1}; \Phi(\tau))^{(1)}$, $H(2; \underline{1}; \Delta)$ and

$$L[\alpha, \beta] = \tilde{S} + \Psi_{\alpha, \beta}(T) \cap L[\alpha, \beta];$$

- (3) $\tilde{S} \cong \mathcal{M}(1, 1)$ and $L[\alpha, \beta] = \tilde{S}$;
- (4) \tilde{S} is a classical Lie algebra of type A_2 , B_2 or G_2 and $L[\alpha, \beta] = \tilde{S}$;
- (5) $\tilde{S} = H(2; (2, 1))^{(2)}$ and $\Psi_{\alpha, \beta}(T) \subset \tilde{S}_p$. Moreover,

$$H(2; (2, 1))^{(2)} \subset L[\alpha, \beta] \subset H(2; (2, 1))^{(2)} \oplus FD_H(x_1^{(p^2)}) \oplus FD_H(x_1^{(p^2-1)}x_2^{(p-1)}).$$

In cases (1), (3), (4) the Lie algebra $L[\alpha, \beta]$ is simple, and $L[\alpha, \beta]^{(1)}$ is simple in all cases.

Proof. If \tilde{S} is not isomorphic to $H(2; (2, 1))^{(2)}$, then the statement follows immediately from [P-St 04, Thm. 4.2]. So assume $\tilde{S} \cong H(2; (2, 1))^{(2)}$. Then [P-St 04, Thm. 4.2] says that $L[\alpha, \beta] \subset \mathcal{G}$ where \mathcal{G} is the p -envelope of $G = H(2; (2, 1))$ in $\text{Der } \tilde{S}$. Recall that $\Psi_{\alpha, \beta}: T + L(\alpha, \beta)_p \rightarrow \text{Der } \tilde{S}$ is a restricted Lie algebra homomorphism. Hence \tilde{S}_p lies in the image of $\Psi_{\alpha, \beta}$. In the present case, $\text{Der } \tilde{S} = \mathcal{G} \oplus F(x_1D_1 + x_2D_2)$; see [B-W 88, Prop. 2.1.8(vii)] for instance. If $\Psi_{\alpha, \beta}(T) \not\subset \mathcal{G}$, then there is a surjective restricted Lie algebra homomorphism $\Psi_{\alpha, \beta}(T + L(\alpha, \beta)_p) \twoheadrightarrow F(x_1D_1 + x_2D_2)$ whose kernel contains \tilde{S}_p . But then [St-F, Lemma 2.4.4(2)] yields that the restricted Lie algebra $\Psi_{\alpha, \beta}(T + L(\alpha, \beta)_p)$ contains 3-dimensional tori, a contradiction. Consequently, $\Psi_{\alpha, \beta}(T + L(\alpha, \beta)_p) \subset \mathcal{G}$, forcing $\Psi_{\alpha, \beta}(T) \subset \mathcal{G}^{[p]} \subset \tilde{S}_p$.

Let \mathfrak{t}' be an optimal 2-dimensional torus in \tilde{S}_p . By [B-W 88, Lemma 1.7.2(b)], there is a torus T' of maximal dimension in $T + L(\alpha, \beta)_p$ such that $\Psi_{\alpha, \beta}(T') = \mathfrak{t}'$. Let H' denote the centralizer of T' in L . Note that $L(\alpha, \beta) = L(\alpha', \beta')$ for some $\alpha', \beta' \in \Gamma(L, T')$ (this follows from the main result of [P 89] and [P-St 99, Cor. 2.10]). Each $i\alpha' + j\beta'$ with $i, j \in \mathbb{F}_p$ can be viewed as a linear function of \mathfrak{t}' .

Since \mathfrak{t}' is optimal, $\mathfrak{t}' \cap \tilde{S} = \mathfrak{t}' \cap \tilde{S}_{(0)}$ is spanned by a nonzero toral element, t_2 say; see [St 92, (VI.1)]. Since $\Gamma(\tilde{S}, \mathfrak{t}') \cup \{0\}$ is a 2-dimensional vector space over \mathbb{F}_p , by Lemma 2.5(4), there is $\delta' \in \Gamma(L(\alpha, \beta), T')$ such that $\delta'(t_2) = 0$. Since, then, δ' also vanishes on $\mathfrak{c}_{\tilde{S}}(\mathfrak{t}')$, the Engel–Jacobson theorem yields that $\tilde{S}(\delta')$ is nilpotent. Since \mathcal{G}/\tilde{S} is solvable, $\mathcal{G}(\delta')$ must be, also. But then $L(\delta')$ is solvable, too. As explained

in [St 92, (VI.4)] the union $\bigcup_{i \in \mathbb{F}_p^\times} \tilde{S}_{i\delta'}$ contains a nonnilpotent element of \mathcal{G} . Hence $\bigcup_{i \in \mathbb{F}_p^\times} L_{i\delta'}$ contains a nonnilpotent element of L_p . Since $L_{i\delta'} \subset \text{rad } L(\delta')$ for all $i \in \mathbb{F}_p^\times$, it follows from [P-St 04, Prop. 3.8] that δ' vanishes on H' .

Recall that $\tilde{S}_p = FD_1^p \oplus \tilde{S}$ and $\mathcal{G} = S_p \oplus V$, where V is the F -span of $D_H(x_1^{(p^2)})$, $D_H(x_2^{(p)})$ and $D_H(x_1^{(p^2-1)}x_2^{(p-1)})$. Hence $\mathcal{G}^3 \subset \tilde{S}$. Pick a toral element $t_1 \in \mathfrak{t}' \setminus \tilde{S}$ (such an element exists by Lemma 2.4). By Lemma 2.7, we may assume that $t_1 = D_1^p + D_1 + D_H(x_1x_2)$ (one should keep in mind here that $\tilde{S}_{(0)}$ is invariant under all automorphisms of S ; see [St 04, Thm. 4.2.6]). Set $V' := (\text{Id} - (\text{ad } t_2)^{p-1})(\text{Id} - (\text{ad } t_1)^{p-1})(V)$. Then

$$\mathfrak{c}_{\tilde{S}}(\mathfrak{t}') \subset \Psi_{\alpha,\beta}(H') \subset \mathfrak{c}_{\mathcal{G}}(\mathfrak{t}') = \mathfrak{c}_{\tilde{S}_p}(\mathfrak{t}') \oplus V', \quad \mathfrak{c}_{\mathcal{G}}(\mathfrak{t}')^3 \subset \mathfrak{c}_{\tilde{S}}(\mathfrak{t}') \subset \Psi_{\alpha,\beta}(H').$$

The elements $(\text{Id} - (\text{ad } t_1)^{p-1})(D_H(x_1^{(p^2)}))$ and $(\text{Id} - (\text{ad } t_1)^{p-1})(D_H(x_1^{(p^2-1)}x_2^{(p-1)}))$ lie in $G_{(p-2)} \subset G_{(1)}$ whereas $[t_1, D_H(x_2^{(p)})] = 0$. Consequently, $(\text{Id} - (\text{ad } t_1)^{p-1})(V) \subset G_{(1)}$. As $\text{ad } t_2$ preserves $G_{(1)}$ we get $V' \subset G_{(1)}$.

We claim that $L[\alpha, \beta] \subset G$. Indeed, suppose the contrary. Recall that $G = \tilde{S} \oplus V' \subsetneq L[\alpha, \beta] + V'$ and $\mathcal{G} = \tilde{S} \oplus FD_1^p \oplus V'$. Then $\mathcal{G} = L[\alpha, \beta] + V'$, hence

$$\mathfrak{t}' \subset \mathfrak{c}_{\mathcal{G}}(\mathfrak{t}') = \mathfrak{c}_{L[\alpha,\beta]+V'}(\mathfrak{t}') = \Psi_{\alpha,\beta}(H') + V'.$$

Since $(\Psi_{\alpha,\beta}(H') + V')^3 \subset \Psi_{\alpha,\beta}(H')$, Jacobson's formula and induction on k enable us to deduce that $(\Psi_{\alpha,\beta}(H') + V')^{[p]^k} \subset (V')^{[p]^k} + \sum_{i=0}^k \Psi_{\alpha,\beta}(H')^{[p]^i}$ for all $k \geq 0$. From our earlier remarks we know that $V' \subset G_{(1)}$ consists of p -nilpotent elements of \mathcal{G} . Therefore, $(\Psi_{\alpha,\beta}(H') + V')^{[p]^e} \subset \sum_{i=0}^e \Psi_{\alpha,\beta}(H')^{[p]^i}$ for all sufficiently large e . Since H' is nilpotent, this forces $\mathfrak{t}' = (\mathfrak{t}')^{[p]^e} \subset (\Psi_{\alpha,\beta}(H'))^{[p]^e}$ for $e \gg 0$. But then δ' vanishes on \mathfrak{t}' . By contradiction, the claim follows.

Suppose $L[\alpha, \beta] \not\subset H(2; (2, 1))^{(2)} \oplus FD_H(x_1^{(p^2)}) \oplus FD_H(x_1^{(p^2-1)}x_2^{(p-1)})$ and pick $\mu \in F^\times$. Recall the elements $t_\mu \in \tilde{S}$ and $h_\mu \in \mathfrak{c}_G(t_\mu)$ from Lemma 2.1. Our present assumption on $L[\alpha, \beta]$ implies that $\mathfrak{c}_{L[\alpha,\beta]}(t_\mu) \not\supseteq C'_\mu$; see Lemma 2.2(i). As $L[\alpha, \beta] \subset G$ by our remarks earlier in the proof, $L[\alpha, \beta]$ contains an element from $(G \cap C_\mu) \setminus C'_\mu$; call it h . In view of Lemma 2.3(ii), we may assume that $h = h_\mu + sD_H(x_1^{(p^2)})$ for some $s \in F$.

Let h_0 denote the p -semisimple part of h in the p -envelope of $L[\alpha, \beta]$ in \mathcal{G} . It is immediate from Lemma 2.3(iv) that the elements h_0 and t_μ are linearly independent. This implies that $\mathfrak{t}_\mu := Fh_0 \oplus Ft_\mu$ is a torus of maximal dimension in \mathcal{G} . Recall that the restricted Lie algebra homomorphism $\Psi_{\alpha,\beta}$ takes $T + L(\alpha, \beta)_p$ into \mathcal{G} . Hence it follows from [St-F, Lemma 2.4.4(2)] that there exists a torus of maximal dimension T'' in L_p contained in $T + L(\alpha, \beta)_p$ and such that $\mathfrak{t}_\mu = \Psi_{\alpha,\beta}(T'')$ and $T \cap \ker \alpha \cap \ker \beta \subset T \cap T''$. We denote by H'' the centralizer of T'' in L . By construction, there exists $\tilde{h} \in H''$ with $\Psi_{\alpha,\beta}(\tilde{h}) = h$.

Set $T_0 := T \cap \ker \alpha \cap \ker \beta$. Because $L(\alpha, \beta) = \mathfrak{c}_L(T_0)$, it is straightforward to see that $L(\gamma'') = L(\alpha, \beta)(\gamma'')$ for every $\gamma'' \in \Gamma(L, T'')$ with $\gamma''(T_0) = 0$. Since $\Psi_{\alpha,\beta}(T'') = \mathfrak{t}_\mu$, there exists $\delta'' \in \Gamma(L, T'')$ such that $\delta''(T_0) = 0$, $\delta''(t_\mu) = 0$ and $\delta''(h_0) \neq 0$; see Lemma 2.5(4). Then $C'_\mu \subset \Psi_{\alpha,\beta}((L(\alpha, \beta))(\delta'')) \subset C_\mu$ and $\delta''(\tilde{h}) \neq 0$. Since $(L(\alpha, \beta))(\delta'') = L(\delta'')$ by the preceding remark, Lemma 2.2(i) shows that δ'' is a solvable root which does not vanish on H'' . In view of [P-St 04, Prop. 3.8], this entails

that every root space $L_{i\delta''} = (\text{rad } L(\delta''))_{i\delta''}$, where $i \in \mathbb{F}_p$, consists of p -nilpotent elements of L_p . Since $\Psi_{\alpha,\beta}$ is a restricted Lie algebra homomorphism, this means that for every $\lambda \in F^\times$ all λ -eigenvectors of the linear operator $(\text{ad } h)|_{C'_\mu}$ must act nilpotently on \tilde{S} . As this contradicts Lemma 2.3(iv), we now derive that our present assumption is false. Thus, $L[\alpha, \beta] \subset H(2; (2, 1))^{(2)} \oplus FD_H(x_1^{(p^2)}) \oplus FD_H(x_1^{(p^2-1)} x_2^{(p-1)})$, completing the proof. \square

If \tilde{S} is a minimal ideal of $L[\alpha, \beta]$ and $TR(\tilde{S}) = 1$, then [P-St 04, Thm. 4.4] implies the following:

Theorem 3.7. *Suppose \tilde{S} is a unique minimal ideal of $L(\alpha, \beta)$ and $TR(\tilde{S}) = 1$. Then there exists $\delta \in \mathbb{F}_p\alpha + \mathbb{F}_p\beta$ such that $\Psi_{\alpha,\beta}(L_\gamma) \subset \tilde{S}$ for all $\gamma \in \Gamma(L, T) \setminus \mathbb{F}_p\delta$. Moreover, one of the following holds:*

- (1) $L[\alpha, \beta] = L[\eta]$ for some $\eta \in \Gamma(L, T) \cap (\mathbb{F}_p\alpha + \mathbb{F}_p\beta)$;
- (2) $\tilde{S} \cong H(2; \underline{1})^{(2)}$, $L[\alpha, \beta] \subset \text{Der } H(2; \underline{1})^{(2)}$ and $\dim \Psi_{\alpha,\beta}(T) = 2$;
- (3) $S \otimes \mathcal{O}(m; \underline{1}) \subset L[\alpha, \beta] \subset (\text{Der } S) \otimes \mathcal{O}(m; \underline{1}) \rtimes (\text{Id} \otimes W(m; \underline{1}))$, where S is one of $\mathfrak{sl}(2)$, $W(1; \underline{1})$, $H(2; \underline{1})^{(2)}$, $\tilde{S} \cong S \otimes \mathcal{O}(m; \underline{1})$, and $m > 0$.

In cases (1) and (2) one can take $\delta = 0$, i.e. $\Psi_{\alpha,\beta}(L_\gamma) \subset \tilde{S}$ for all $\gamma \in \Gamma(L, T)$.

More information on the two-sections of L can be found in [P-St 04, Sect. 4].

4. Nonstandard tori of maximal dimension

From now on we assume that T is a nonstandard torus of maximal dimension in the semisimple p -envelope L_p of L . In light of [P 94, Thm. 1] this implies that $p = 5$. As explained in Sect. 2, the linear functions on T can be regarded as functions on the nilpotent restricted Lie algebra $\mathfrak{c}_{L_p}(T)$. Set $H := \mathfrak{c}_L(T)$ and define

$$\Omega = \Omega(L, T) := \{\delta \in \Gamma(L, T) \mid \delta(H^3) \neq 0\}.$$

As T is a torus of maximal dimension in L_p , it is immediate from [P 94, Thm. 1(ii)] that there exist \mathbb{F}_p -independent roots $\alpha, \beta \in \Gamma(L, T)$ for which $L[\alpha, \beta] \cong \mathcal{M}(1, 1)$. By Lemmas 4.1 and 4.4 of [P 94], we then have $i\alpha + j\beta \in \Omega$ for all nonzero $(i, j) \in \mathbb{F}_p^2$. In particular, $\Omega \neq \emptyset$. In view of Schue's lemma [St 04, Prop. 1.3.6(1)], this yields

$$(4.1) \quad L_\gamma = \sum_{\delta \in \Omega} [L_\delta, L_{\gamma-\delta}] \quad (\forall \gamma \in \Gamma(L, T) \cup \{0\}).$$

Because of [P 94, Thm. 1(ii)] we can also assume that $TR(L) \geq 3$. Our main goal in this section is to give a preliminary description of the 2-sections of L relative to T . More precisely, we will go through all possible types of 2-sections (described in Sect. 3) and eliminate some of them by using our assumption on T .

Lemma 4.1. *For any nonsolvable $\alpha \in \Omega$ there exists $\beta \in \Gamma(L, T)$ such that $L[\alpha, \beta] \cong \mathcal{M}(1, 1)$ and $\alpha([L_{i\alpha}, L_{-i\alpha}], [L_\beta, L_{-\beta}]) \neq 0$ for some $i \in \mathbb{F}_p^\times$.*

Proof. Since α is nonsolvable and $\alpha(H^3) \neq 0$, Theorem 3.3 implies that $L[\alpha] \cong H(2; \underline{1})^{(2)} \oplus F(1 + x_1)^4 \partial_1$. By [P-St 04, Thm. 3.5], there is $k \in \mathbb{F}_p^\times$ for which the set $\Omega_1 := \{\delta \in \Gamma(L, T) \mid \delta([L_{k\alpha}, L_{-k\alpha}]) \neq 0\}$ is nonempty. Since $\Psi_\alpha(H) \cap H(2; \underline{1})^{(2)}$ has

codimension one in $\Psi_\alpha(H)$, Schue's lemma [St 04, Prop. 1.3.6(1)] implies that there exists $\beta \in \Omega_1$ with the property that

$$\Psi_\alpha(H) = \Psi_\alpha(H) \cap H(2; \underline{1})^{(2)} + \Psi_\alpha([L_\beta, L_{-\beta}]).$$

Hence there exist $h_1 \in L(\alpha)^{(\infty)} \cap H$ and $h_2 \in [L_\beta, L_{-\beta}]$ with $\alpha([h_2, [h_2, h_1]]) \neq 0$. Note that $\beta([h_2, [h_2, h_1]]) \in \beta([L_\beta, L_{-\beta}]^2) = 0$ by Lemma 3.1. In particular, α and β are linearly independent over \mathbb{F}_p . Since $\beta \in \Omega_1$, we then have

$$(4.2) \quad \beta([h_2, [h_2, h_1]]) = 0; \quad \alpha([h_2, [h_2, h_1]]) \neq 0; \quad \beta([L_{k\alpha}, L_{-k\alpha}]) \neq 0.$$

We now look more closely at the T -semisimple quotient $L[\alpha, \beta]$ of the 2-section $L(\alpha, \beta)$. Since α is nonsolvable, $L[\alpha, \beta] \neq (0)$. Let \tilde{S} denote the p -envelope of the T -socle \tilde{S} of $L[\alpha, \beta]$ in $\text{Der } \tilde{S}$, and set $u := \Psi_{\alpha, \beta}([h_2, [h_2, h_1]])$. Given $x \in \tilde{S}$ we write x_s for the p -semisimple part of x in \tilde{S} . Because the roots α, β are \mathbb{F}_p -independent, $h_1 \in L(\alpha)^{(\infty)} \cap H = \sum_{j \in \mathbb{F}_p^\times} [L_{j\alpha}, L_{-j\alpha}]$ and $h_2 \in [L_\beta, L_{-\beta}]$, it follows from Theorems 3.3, 3.5, 3.6 and 3.7 that $u \in \tilde{S}$. Now relations (4.2) enable us to find $v \in \tilde{S} \cap \Psi_{\alpha, \beta}([L_{k\alpha}, L_{-k\alpha}])$ such that the span of u_s and v_s is 2-dimensional. This yields $\Psi_{\alpha, \beta}(T) \subset \tilde{S}$ showing that $TR(\tilde{S}) = 2$. Since $\beta([L_{k\alpha}, L_{-k\alpha}]) \neq 0$, we also deduce that there are \mathbb{F}_p -independent $\delta_1, \delta_2 \in \Gamma(L, T)$ for which $[\Psi_{\alpha, \beta}(L_{\delta_1}), \Psi_{\alpha, \beta}(L_{\delta_2})] \neq 0$. In view of Theorem 3.5, this implies that \tilde{S} is a minimal ideal of $L[\alpha, \beta]$.

Theorem 3.6 now says that \tilde{S} is a simple Lie algebra and $\Psi_{\alpha, \beta}(L_\gamma) \subset \tilde{S}$ for all $\gamma \in \Gamma(L, T) \cap (\mathbb{F}_p\alpha + \mathbb{F}_p\beta)$. Since $\alpha(H, [L_{k\alpha}, L_{-k\alpha}]) \neq 0$, the torus $\Psi_{\alpha, \beta}(T) \subset \tilde{S} = \tilde{S}_p$ is nonstandard. Applying [P 94, Thm. 1(ii)] we conclude that $L[\alpha, \beta] \cong \mathcal{M}(1, 1)$, finishing the proof. \square

Proposition 4.2. *If $\alpha \in \Omega$ and $\beta \in \Gamma(L, T)$, then one of the following occurs:*

- 1) $L[\alpha, \beta] = (0)$.
- 2) $L[\alpha, \beta] = L[\delta]$ for some $\delta \in \Gamma(L, T)$.
- 3) $L[\delta_1]^{(1)} \oplus L[\delta_2]^{(1)} \subset L[\alpha, \beta] \subset L[\delta_1] \oplus L[\delta_2]$ for some $\delta_1, \delta_2 \in \Gamma(L, T)$.
- 4) $S \otimes \mathcal{O}(m; \underline{1}) \subset L[\alpha, \beta] \subset (\text{Der } S) \otimes \mathcal{O}(m; \underline{1}) \rtimes (\text{Id} \otimes W(m; \underline{1}))$, where S is one of $\mathfrak{sl}(2)$, $W(1; \underline{1})$, $H(2; \underline{1})^{(2)}$, $\tilde{S} \cong S \otimes \mathcal{O}(m; \underline{1})$, and $m > 0$.
- 5) $H(2; (2, 1))^{(2)} \subset L[\alpha, \beta] \subset H(2; (2, 1))$ and $\tilde{S} = H(2; (2, 1))^{(2)} = L[\alpha, \beta]^{(1)}$. Furthermore, each $\eta \in \Gamma(L[\alpha, \beta], \Psi_{\alpha, \beta}(T))$ is Hamiltonian, $\eta(\Psi_{\alpha, \beta}(T) \cap \tilde{S}) \neq 0$, and $\Gamma(L[\alpha, \beta], \Psi_{\alpha, \beta}(T)) = (\mathbb{F}_p\alpha \oplus \mathbb{F}_p\beta) \setminus \{0\}$.
- 6) $L[\alpha, \beta] \cong \mathcal{M}(1, 1)$.

Proof. (a) Set $\overline{T} := \Psi_{\alpha, \beta}(T)$ and $\overline{H} := \Psi_{\alpha, \beta}(H)$. If $\Gamma(L[\alpha, \beta], \overline{T}) = \emptyset$, then $L(\alpha, \beta)$ is solvable, forcing $L[\alpha, \beta] = (0)$. If $\emptyset \neq \Gamma(L[\alpha, \beta], \overline{T}) \subset \mathbb{F}_p\delta$ for a single root δ , then for any $\delta' \in (\mathbb{F}_p\alpha \oplus \mathbb{F}_p\beta) \setminus \mathbb{F}_p\delta$ we have that $L_{\delta'} \subset \text{rad}_T L(\alpha, \beta)$. Then $L[\alpha, \beta] = L[\delta]$. So we may assume from now that $\Gamma(L[\alpha, \beta], \overline{T})$ contains two roots independent over \mathbb{F}_p . Then $L[\alpha, \beta]$ is described in Theorems 3.5, 3.6 and 3.7. Let \tilde{S} be the T -socle of $L[\alpha, \beta]$. If \tilde{S} is not a minimal ideal of $L[\alpha, \beta]$, then Theorem 3.5 says that we are in case 3) of this proposition. Thus, we may assume further that \tilde{S} is a minimal ideal of $L[\alpha, \beta]$.

(b) Suppose $TR(\tilde{S}) = 2$. Then $L[\alpha, \beta]$ is described in Theorem 3.6. Since $\alpha(H^3) \neq 0$, there exists $\eta \in \Gamma(\tilde{S}, \bar{T})$ with $\eta(\bar{H}^3) \neq 0$. In cases (1)–(4) of Theorem 3.6 we have $\bar{H}^3 \subset (\bar{T} + \bar{H} \cap \tilde{S})^3 = (\bar{H} \cap \tilde{S})^3$, implying that $\bar{H}' = \mathfrak{c}_{\tilde{S}}(\bar{T})$ acts nontriangulably on \tilde{S} . But then [P 94, Thm. 1(ii)] shows that $\tilde{S} \cong \mathcal{M}(1, 1)$. This brings up case 6) of this proposition.

(c) Suppose $L[\alpha, \beta]$ is as in case (5) of Theorem 3.6. Then $\tilde{S} \cong H(2; (2, 1))^{(2)}$ and $L[\alpha, \beta] \subset H(2; (2, 1))^{(2)} \oplus FD_H(x_1^{(p^2)}) \oplus FD_H(x_1^{(p^2-1)} x_2^{(p-1)})$. Furthermore, $\bar{T} \subset \tilde{S}_p$. If no root in $\Gamma(\tilde{S}, \bar{T})$ vanishes on $\bar{T} \cap \tilde{S}$, then Lemma 2.5(2) shows that we are in case 5) of this this proposition. So assume for a contradiction that there is $\delta \in \Gamma(\tilde{S}, \bar{T})$ with $\delta(\bar{T} \cap \tilde{S}) = 0$. By Lemma 2.4, we have $\bar{T} \cap \tilde{S} = Fu_2 \neq (0)$. Since δ vanishes on $u_2 \in \bar{T} \cap \tilde{S}$, we may assume without loss that u_2 is a toral element. As before, we put $G = H(2; (2, 1))$ and $\mathcal{G} = \tilde{S}_p \oplus V$, where $V \subset \text{Der } \tilde{S}$ is defined in Sect. 2. Since $\alpha \in \Omega$, the Lie algebra \bar{H}^3 acts nonnilpotently on S .

(c1) We first suppose that $\bar{T} \cap \tilde{S} \not\subset S_{(0)}$. Then we can find $\Psi_{\alpha, \beta}$ such that $\bar{T} \cap \tilde{S} = Ft_\mu$ where $\mu \in F$; see Lemma 2.1. Thus, no generality will be lost by assuming that $u_2 = t_\mu$. But then it follows from Lemma 2.2(i) that

$$\bar{H} \subset C_\mu \cap \left(H(2; (2, 1))^{(2)} \oplus FD_H(x_1^{(p^2)}) \oplus FD_H(x_1^{(p^2-1)} x_2^{(p-1)}) \right) = C'_\mu$$

and $[\bar{H}, \bar{H}] \subset [C'_\mu, C'_\mu] = (0)$. Since \bar{H} acts nontriangulably on \tilde{S} , this is impossible.

(c2) Now suppose that $\bar{T} \cap \tilde{S} \subset S_{(0)}$. Then $\bar{T} \cap \bar{S}_{(0)}$ contains a nonzero p -semisimple element, say t ; see Lemma 2.4. It follows from Lemma 2.4 and our earlier remarks that $\mathcal{G} = \bar{T} + G$. As $\text{gr } t \in G_{(0)}/G_{(1)} \cong \mathfrak{sl}(2)$ acts invertibly on $G_{(-1)} = G/G_{(0)}$, this implies that $\bar{H} \subset \bar{T} + \mathfrak{c}_G(\bar{T}) = \bar{T} + \mathfrak{c}_{G_{(1)}}(\bar{T})$. But then $\bar{H}^{(1)} \subset G_{(1)}$ acts nilpotently on G , a contradiction.

As a result, no root in $\Gamma(\tilde{S}, \bar{T})$ vanishes on $\bar{H} \cap \tilde{S}$ and we are in case 5) of this proposition; see Lemma 2.5(2).

(d) If $L[\alpha, \beta]$ is as in case (1) of Theorem 3.7, then it is listed in the present proposition as type 2). If $L[\alpha, \beta]$ is as in case (2) of Theorem 3.7, then $\tilde{S} = H(2; \underline{1})^{(2)}$, $L[\alpha, \beta] \subset \text{Der } H(2; \underline{1})^{(2)}$, and \bar{T} is a 2-dimensional torus in $\text{Der } \tilde{S}$. It is well-known that any 2-dimensional torus in $\text{Der } \tilde{S}$ is self-centralizing; see [St 92, (III.1)] for instance. But then $\gamma(H^{(1)}) = 0$ for all $\gamma \in \mathbb{F}_p\alpha \oplus \mathbb{F}_p\beta$. Thus, this case cannot occur in our situation. Finally, case (3) of Theorem 3.7 is listed as type 4) in the present proposition. \square

Corollary 4.3. *Let $\alpha \in \Omega$ and $\beta \in \Gamma(L, T)$. If $L[\alpha, \beta]$ is as in cases 1), 2), 3), 5) or 6) of Proposition 4.2, then $\sum_{i \in \mathbb{F}_p^\times} (\text{rad } L(\gamma))_{i\gamma} \subset \text{rad}_T L(\alpha, \beta)$ for all nonzero $\gamma \in \mathbb{F}_p\alpha + \mathbb{F}_p\beta$.*

Proof. If $L[\alpha, \beta]$ is of type 1) or 2), then all 1-sections of $L[\alpha, \beta]$ are semisimple and there is nothing to prove. If $L[\alpha, \beta]$ is of type 3), then there are $h_i \in \bar{H} \cap L[\delta_i]$ such that $\delta_i(h_i) \neq 0$, where $i = 1, 2$ (recall that $\bar{H} = \Psi_{\alpha\beta}(H)$). It follows that $\text{rad } L[\alpha, \beta](\delta_i) \subset \bar{H} + L[\delta_i]^{(1)}$. As each $L[\delta_i]^{(1)}$ is simple, we get $\text{rad}(L[\alpha, \beta](\gamma)) \subset \bar{H}$ for all nonzero $\gamma \in \mathbb{F}_p\alpha \oplus \mathbb{F}_p\beta$. If $L[\alpha, \beta]$ is of type 5) or 6), then all T -roots of $L[\alpha, \beta]$ are Hamiltonian and the corresponding root spaces are 5-dimensional (see Lemma 2.5

and [P 94, Lemmas 4.1 & 4.4]). Hence in these cases $\text{rad}(L[\alpha, \beta](\gamma)) \subset \overline{H}$ for all $\gamma \in (\mathbb{F}_p\alpha \oplus \mathbb{F}_p\beta) \setminus \{0\}$. \square

Lemma 4.4. *The following hold for every $\gamma \in \Gamma(L, T)$ with $\gamma(H) \neq 0$:*

- (a) *All elements in $\bigcup_{i \in \mathbb{F}_p^\times} (H^3 \cap [(\text{rad } L(\gamma))_{i\gamma}, L_{-i\gamma}])$ are p -nilpotent in L_p .*
- (b) *If $\gamma \in \Omega$, then all elements in $\bigcup_{i \in \mathbb{F}_p^\times} ((\text{rad } L(\gamma))_{i\gamma} \cup [(\text{rad } L(\gamma))_{i\gamma}, L_{-i\gamma}])$ are p -nilpotent in L_p .*

Proof. We will treat both cases simultaneously. Set

$$\begin{aligned}\Omega' &:= \left\{ \alpha \in \Gamma(L, T) \mid \alpha \left(\bigcup_{i \in \mathbb{F}_p^\times} (H^3 \cap [(\text{rad } L(\gamma))_{i\gamma}, L_{-i\gamma}]) \right) \neq 0 \right\}, \\ \Omega'' &:= \left\{ \alpha \in \Gamma(L, T) \mid \alpha \left(\bigcup_{i \in \mathbb{F}_p^\times} ((\text{rad } L(\gamma))_{i\gamma}^{[p]} \cup [(\text{rad } L(\gamma))_{i\gamma}, L_{-i\gamma}]) \right) \neq 0 \right\}.\end{aligned}$$

Assume for a contradiction that either $\Omega' \neq \emptyset$ or $\gamma \in \Omega$ and $\Omega'' \neq \emptyset$. Note that $\Omega' \subset \Omega'' \cap \Omega$. Since $\gamma(H) \neq 0$, Schue's lemma [St 04, Prop. 1.3.6(1)] shows that there exists $\mu \in \Omega'$ or $\mu \in \Omega''$ for $\gamma \in \Omega$ such that

$$(4.3) \quad \gamma([L_\mu, L_{-\mu}]) \neq 0.$$

In both cases, the type of $L[\gamma, \mu]$ is determined by Proposition 4.2. If $L[\gamma, \mu]$ is as in cases 1), 2), 3), 5) or 6) of Proposition 4.2, then $\sum_{i \in \mathbb{F}_p^\times} (\text{rad } L(\gamma))_{i\gamma} \subset \text{rad}_T L(\gamma, \mu)$ by Corollary 4.3. Since $\mu \in \Omega''$ in both cases, this yields $L_{\pm\mu} \subset \text{rad}_T L(\gamma, \mu)$. Easy induction on n based on (4.3) now gives

$$\sum_{i \in \mathbb{F}_p^\times} (\text{rad } L(\gamma))_{i\gamma} \subset \bigcap_{n \geq 1} (\text{rad}_T L(\gamma, \mu))^{(n)} = (0).$$

Since this contradicts our assumption that either Ω' or Ω'' is nonempty, $L[\gamma, \mu]$ must be of type 4). Then the minimal ideal of $L[\gamma, \mu]$ has the form $\tilde{S} = S \otimes \mathcal{O}(m; \underline{1})$, where S is a restricted simple Lie algebra of absolute toral rank 1 and $m > 1$. According to [P-St 99, Thm. 3.2] we can choose $\Psi_{\gamma, \mu}$ such that $\overline{T} = \Psi_{\gamma, \mu}(T)$ has the form $F(h_0 \otimes 1) \oplus F(d \otimes 1 + \text{Id}_S \otimes t_0)$ for some $d \in \text{Der } S$ and some nonzero toral elements $t_0 \in W(m; \underline{1})$ and $h_0 \in S$.

Since $TR(L[\gamma, \mu]) = 2$, the roots γ and μ span the dual space of \overline{T} . Therefore, $\gamma(h_0 \otimes 1) \neq 0$ or $\mu(h_0 \otimes 1) \neq 0$. It is straightforward to see that γ vanishes on all $(\text{rad } L(\gamma))_{i\gamma}^{[p]}$ and $[(\text{rad } L(\gamma))_{i\gamma}, L_{-i\gamma}]$ with $i \in \mathbb{F}_p^\times$. Because $\mu \in \Omega''$, this observation in conjunction with (4.3) shows that $\Psi_{\gamma, \mu}(L_{i\gamma+j\mu}) \subset S \otimes \mathcal{O}(m; \underline{1})$ for all nonzero $(i, j) \in (\mathbb{F}_p)^2$. There are in both cases

$$x \in \bigcup_{i \in \mathbb{F}_p^\times} ((\text{rad } L(\gamma))_{i\gamma}^{[p]} \cup [(\text{rad } L(\gamma))_{i\gamma}, L_{-i\gamma}]) \quad \text{and} \quad h \in [L_\mu, L_{-\mu}]$$

such that $\gamma(x^{[p]}) = 0$, $\mu(x^{[p]}) \neq 0$ and $\gamma(h) \neq 0$. But then $2 \leq TR(S \otimes \mathcal{O}(m; \underline{1})) = TR(S) = 1$, a contradiction. \square

Proposition 4.5. *Let $\alpha \in \Omega$ and $\beta \in \Gamma(L, T)$ be such that $L[\alpha, \beta]$ is as in case 4) of Proposition 4.2. Then $\tilde{S} \cong S \otimes \mathcal{O}(1; \underline{1})$, where $S = H(2; \underline{1})^{(2)}$, and $\Psi_{\alpha, \beta}$ can be chosen such that $\overline{T} := \Psi_{\alpha, \beta}(T) = F(h_0 \otimes 1) \oplus F(\text{Id}_S \otimes (1 + x_1)\partial_1)$ for some*

nonzero toral element $h_0 \in S$. Furthermore, $\Omega \neq \Gamma(L, T)$ and the following hold for $\gamma \in \Gamma(L[\alpha, \beta], \overline{T})$:

$$\begin{aligned} \gamma \in \Omega &\Leftrightarrow \gamma(h_0 \otimes 1) \neq 0; \\ \gamma \notin \Omega &\Rightarrow \alpha(L_\gamma^{[p]}) \neq 0 \text{ or } \beta(L_\gamma^{[p]}) \neq 0. \end{aligned}$$

Proof. By our assumption, $\tilde{S} = S \otimes \mathcal{O}(m; \underline{1})$ where $m \geq 1$, S is one of $\mathfrak{sl}(2)$, $W(1; \underline{1})$, $H(2; \underline{1})^{(2)}$. Recall that $\Psi_{\alpha, \beta}$ takes $T + L(\alpha, \beta)_p$ into $\text{Der}(S \otimes \mathcal{O}(m; \underline{1}))$. Let

$$\pi: \text{Der}(S \otimes \mathcal{O}(m; \underline{1})) = (\text{Der } S) \otimes \mathcal{O}(m; \underline{1}) \rtimes (\text{Id}_S \otimes W(m; \underline{1})) \rightarrow W(m; \underline{1})$$

denote the canonical projection. According to [P-St 99, Thm. 3.2], we can choose $\Psi_{\alpha, \beta}$ such that

$$\overline{T} := \Psi_{\alpha, \beta}(T) = F(h_0 \otimes 1) \oplus F(d \otimes 1 + \text{Id}_S \otimes t_0),$$

where Fh_0 is a maximal torus of S , $d \in \text{Der } S$ and t_0 is a toral element of $W(m; \underline{1})$. Moreover, if $t_0 \in W(m; \underline{1})_{(0)}$, then $t_0 = \sum_{i=1}^m s_i x_i \partial_i$, where $s_i \in \mathbb{F}_p$, and if $t_0 \notin W(m; \underline{1})_{(0)}$, then $d = 0$ and $t_0 = (1 + x_1) \partial_1$.

Our argument is quite long and will be split into two parts, each part consisting of several intermediate statements. Given a subset X of $T + L(\alpha, \beta)_p$ we denote by \overline{X} the set $\{\Psi_{\alpha, \beta}(x) \mid x \in X\}$. If $\{x_1, \dots, x_m\}$ is a generating set of the maximal ideal $\mathcal{O}(m; \underline{1})_{(1)}$, then we sometimes invoke the notation $\mathcal{O}(m; \underline{1}) = F[x_1, \dots, x_m]$.

Part A. We first consider the case where $t_0 \in W(m; \underline{1})_{(0)}$.

Claim 1. $\pi(\overline{H}) \subset W(m; \underline{1})_{(0)}$.

Indeed, suppose the contrary. Then Schue's lemma [St 04, Prop. 1.3.6(1)] shows that there exists $\kappa \in \Gamma(L, T)$ with $\kappa(H) \neq 0$ such that $\pi(\overline{[L_\kappa, L_{-\kappa}]}) \not\subset W(m; \underline{1})_{(0)}$. Then there is $E \in [L_\kappa, L_{-\kappa}]$ such that $\overline{E} = \overline{E}' + \text{Id}_S \otimes \pi(\overline{E})$ with $\overline{E}' \in (\text{Der } S) \otimes \mathcal{O}(m; \underline{1})$ and $\pi(\overline{E}) \equiv \sum_{i=1}^m a_i \partial_i \not\equiv 0 \pmod{W(m; \underline{1})_{(0)}}$ for some $a_i \in F$. No generality will be lost by assuming that $a_1 \neq 0$. Then

$$0 = [t_0, \pi(\overline{E})] \equiv \sum_{i=1}^m a_i s_i \partial_i \pmod{W(m; \underline{1})_{(0)}},$$

forcing $s_1 = 0$. But then $h_0 \otimes x_1^{p-1} \in \overline{H}$ and

$$(\text{ad } \overline{E})^{p-1}(h_0 \otimes x_1^{p-1}) \in F^\times(h_0 \otimes 1) + S \otimes \mathcal{O}(m; \underline{1})_{(1)},$$

which implies that $[L_\kappa, L_{-\kappa}]^3 \not\subset \text{nil } H_p$. As this contradicts Lemma 3.1, the claim follows. \checkmark

Claim 2. There exists $\nu \in \Gamma(L[\alpha, \beta], \overline{T})$ with $\pi(\overline{L}_\nu) \not\subset W(m; \underline{1})_{(0)}$ and $\nu(h_0 \otimes 1) = 0$.

Indeed, \tilde{S} is derivation simple and $\pi(\overline{T} + \overline{H}) \subset W(m; \underline{1})_{(0)}$ by our general assumption in this part and Claim 1. Hence there is $\nu \in \Gamma(L[\alpha, \beta], \overline{T})$ with $\pi(\overline{L}_\nu) \not\subset W(m; \underline{1})_{(0)}$. Since $\pi([h_0 \otimes 1, \overline{L}_\nu]) = 0$, it must be that $\nu(h_0 \otimes 1) = 0$. \checkmark

Claim 3. If $\gamma \in \Gamma(L[\alpha, \beta], \overline{T})$, then $\gamma \in \Omega \Leftrightarrow \gamma(h_0 \otimes 1) \neq 0$.

Let γ be any root in $\Gamma(L[\alpha, \beta], \overline{T})$ with $\gamma(h_0 \otimes 1) = 0$. As $h_0 \otimes 1 \in \overline{T}$ is a nonzero toral element, $\gamma \in \mathbb{F}_p^\times \nu$, where ν is the root from Claim 2. Hence there is $\overline{E} \in \overline{L}_{i\gamma}$ for some $i \in \mathbb{F}_p^\times$, such that $\pi(\overline{E}) \not\subset W(m; \underline{1})_{(0)}$. As before, we have that $\pi(\overline{E}) \equiv \sum_{i=1}^m a_i \partial_i \not\equiv 0 \pmod{W(m; \underline{1})_{(0)}}$, and it can be assumed that $a_1 \neq 0$. Then $h_0 \otimes x_1 \in \tilde{S}_{-i\gamma}$. Note

that $h_0 \otimes \mathcal{O}(m; \underline{1})$ is an abelian ideal of the centralizer of $h_0 \otimes 1$ in $\text{Der } \tilde{S}$. Consequently, $h_0 \otimes x_1 \in \text{rad}(L[\alpha, \beta](\gamma))_{-i\gamma}$ and

$$a_1 h_0 \otimes 1 \equiv [\overline{E}, h_0 \otimes x_1] \pmod{S \otimes \mathcal{O}(m; \underline{1})_{(1)}}.$$

It follows that $[L_{i\gamma}, (\text{rad } L(\gamma))_{-i\gamma}]$ contains an element which is not p -nilpotent in L_p . Then $\gamma \notin \Omega$ by Lemma 4.4. Since $\alpha \in \Omega$, these considerations show that $\alpha(h_0 \otimes 1) \neq 0$. As a consequence,

$$i\alpha + j\gamma \in \Omega \Leftrightarrow (i\alpha + j\gamma)(H^3) \neq 0 \Leftrightarrow i \in \mathbb{F}_p^\times \Leftrightarrow (i\alpha + j\gamma)(h_0 \otimes 1) \neq 0,$$

hence the claim. \checkmark

Claim 4. The Lie algebra $\pi(\overline{H})^3$ consists of p -nilpotent elements of $W(m; \underline{1})$.

Otherwise, there is $y \in \overline{H}^3$ with $y^{[p]^e} \in \overline{T} \setminus F(h_0 \otimes 1)$, so that $y^{[p]^e} = b_1(h_0 \otimes 1) + b_2(d \otimes 1 + \text{Id}_S \otimes t_0)$ for some $b_1 \in F$ and $b_2 \in F^\times$. Let $\nu \in \Gamma(L[\alpha, \beta], \overline{T})$ be as in Claim 2. Then $\nu(h_0 \otimes 1) = 0$ and $\nu(d \otimes 1 + \text{Id}_S \otimes t_0) \neq 0$, forcing $\nu(y^{[p]^e}) \neq 0$. It follows that $\nu \in \Omega$. This contradicts Claim 3, however. \checkmark

Claim 5. $d \in Fh_0$.

Claim 1 in conjunction with our standing hypothesis in this part shows that there is a Lie algebra homomorphism

$$\Psi: (\text{Der } S) \otimes \mathcal{O}(m; \underline{1}) + (\overline{H} + \overline{T}) \longrightarrow \text{Der } S$$

whose kernel is spanned by $(\text{Der } S) \otimes \mathcal{O}(m; \underline{1})_{(1)}$ and those elements of $\overline{H} + \overline{T}$ which map $(\text{Der } S) \otimes \mathcal{O}(m; \underline{1})$ into $(\text{Der } S) \otimes \mathcal{O}(m; \underline{1})_{(1)}$. Suppose $d \notin Fh_0$. Then $\Psi(\overline{T}) = Fh_0 \oplus Fd$. Since d is a semisimple derivation of S , it follows that $S = H(2; \underline{1})^{(2)}$ and $\Psi(\overline{T})$ is a torus of maximal dimension in $\text{Der } S$. Since every such torus is self-centralizing in $\text{Der } S$, by [St 92, (III.1)], it must be that $\overline{H} \subset \overline{T} + \ker \Psi$. Note that

$$(\overline{H} + \overline{T}) \subset (\text{Der } S) \otimes \mathcal{O}(m; \underline{1}) + F(\text{Id}_S \otimes t_0) + \text{Id}_S \otimes \pi(\overline{H})$$

and $F(\text{Id}_S \otimes t_0) + \text{Id}_S \otimes \pi(\overline{H}) \subset \ker \Psi$ by our assumption on t_0 and Claim 1. Hence

$$\begin{aligned} \overline{H} &\subset (\overline{T} + \ker \Psi) \cap \overline{H} \subset (\ker \Psi) \cap (\overline{H} + \overline{T}) + \overline{T} \\ &\subset (\text{Der } S) \otimes \mathcal{O}(m; \underline{1})_{(1)} + F(\text{Id}_S \otimes t_0) + \text{Id}_S \otimes \pi(\overline{H}) + \overline{T}, \end{aligned}$$

forcing $\overline{H}^3 \subset (\text{Der } S) \otimes \mathcal{O}(m; \underline{1})_{(1)} + \text{Id}_S \otimes \pi(\overline{H})^3$. In view of Claim 4 the Lie algebra on the right acts nilpotently on $S \otimes \mathcal{O}(m; \underline{1})$. But then \overline{H}^3 acts nilpotently on $L[\alpha, \beta]$, a contradiction. \checkmark

As a consequence, $\overline{H} \cap (S \otimes \mathcal{O}(m; \underline{1})) = \mathfrak{c}_S(h_0) \otimes \text{Ann}_{\mathcal{O}(m; \underline{1})}(t_0)$ and we may take $d = 0$.

Claim 6. Let ν be as in Claim 2. Then

$$\overline{H} \cap \tilde{S} \subset \Psi_{\alpha, \beta}([(\text{rad } L(\nu))_{-\nu}, L_\nu]) + \overline{H} \cap (S \otimes \mathcal{O}(m; \underline{1})_{(1)}).$$

By definition, there is $\overline{E} \in \overline{L}_\nu$ such that

$$\pi(\overline{E}) \equiv \sum_{i=1}^m a_i \partial_i \not\equiv 0 \pmod{W(m; \underline{1})_{(0)}}, \quad a_1 \neq 0.$$

We have shown in the course of the proof of Claim 3 that $\mathfrak{c}_S(h_0) \otimes x_1 \subset \overline{\text{rad } L(\nu)}_{-\nu}$. Then $\mathfrak{c}_S(h_0) \otimes F \subset [\overline{E}, \tilde{S}_{-\nu}] + \overline{H} \cap (\tilde{S} \cap \mathcal{O}(m; \underline{1})_{(1)})$. As a consequence,

$$\begin{aligned} \overline{H} \cap \tilde{S} &= \mathfrak{c}_S(h_0) \otimes \text{Ann}_{\mathcal{O}(m; \underline{1})}(t_0) \subset \mathfrak{c}_S(h_0) \otimes F + \mathfrak{c}_S(h_0) \otimes \mathcal{O}(m; \underline{1})_{(1)} \\ &\subset [\overline{L_\nu}, \overline{(\text{rad } L(\nu))_{-\nu}}] + \overline{H} \cap (S \otimes \mathcal{O}(m; \underline{1})_{(1)}). \quad \checkmark \end{aligned}$$

Claim 7. If ν is as in Claim 2, then $\nu(H) = 0$.

As $S \otimes F$ is \overline{T} -stable and S is not nilpotent, there is $\mu \in \Gamma(\tilde{S}, \overline{T})$ with $(S \otimes F)_\mu \neq (0)$. Then $\mu(\text{Id}_S \otimes t_0) = 0$ and hence $\mu(h_0 \otimes 1) \neq 0$. It follows that

$$L[\alpha, \beta](\mu) \subset S \otimes \mathcal{O}(m; \underline{1}) + \overline{H} \subset (\text{Der } S) \otimes \mathcal{O}(m; \underline{1}) + \text{Id}_S \otimes W(m; \underline{1})_{(0)}.$$

Let $\Phi: L[\alpha, \beta](\mu) \rightarrow \text{Der } S$ denote the natural \overline{T} -equivariant Lie algebra homomorphism with $\ker \Phi = L[\alpha, \beta](\mu) \cap ((\text{Der } S) \otimes \mathcal{O}(m; \underline{1})_{(1)} + \text{Id}_S \otimes W(m; \underline{1})_{(0)})$ and $S \subset \text{im } \Phi$. Then [St 04, Thms 1.2.8 & 1.3.11] show that

$$TR(\ker \Phi) \leq TR(L[\alpha, \beta](\mu)) - TR(S) \leq TR(L(\mu)) - TR(S) \leq 1 - TR(S) \leq 0,$$

implying that $\ker \Phi$ is a nilpotent ideal of $L[\alpha, \beta](\mu)$. As $\Phi(L[\alpha, \beta](\mu))$ contains S , it is semisimple, hence isomorphic to $L[\mu]$. Note that $\mu \in \Omega$ by Claim 3. As $L[\mu] \neq (0)$, Theorem 3.3 says that $p = 5$ and $\Phi(L[\alpha, \beta](\mu)) \cong H(2; \underline{1})^{(2)} \oplus F(1 + x_1)^4 \partial_2$. In particular, μ is Hamiltonian. Observe that

$$(4.4) \quad (\text{ad } (1 + x_1)^4 \partial_2)^2 (D_H((1 + x_1)^3 x_2^3)) = D_H((1 + x_1)x_2).$$

By (the proof of) Lemma 3.4, we may assume that $h_0 = D_H((1 + x_1)x_2)$. Then (4.4) shows that there exists $\mathcal{D} \in \Phi(\overline{H})$ such that $[\mathcal{D}, [\mathcal{D}, \mathfrak{c}_S(h_0)]] \not\subset \text{nil } \mathfrak{c}_S(h_0)$.

Note that $\text{nil } \mathfrak{c}_S(h_0)$ has codimension 1 in $\mathfrak{c}_S(h_0)$. As $\ker \Phi$ acts nilpotently on $L[\alpha, \beta](\mu)$, there is $\tilde{\mathcal{D}} \in \overline{H}$ with $\mu([\tilde{\mathcal{D}}, [\tilde{\mathcal{D}}, \tilde{S} \cap \overline{H}]]) \neq 0$. Since $\overline{H} \cap (S \otimes \mathcal{O}(m; \underline{1})_{(1)})$ is an ideal of \overline{H} , Claim 6 entails that $\Psi_{\alpha, \beta}((\text{rad } L(\nu))_{-\nu}, L_\nu) \cap \overline{H}^3$ does not consist of p -nilpotent elements of L_p . In view of Lemma 4.4(1), this yields that $\nu(H) = 0$. \checkmark

Since $t_0 \in W(m; \underline{1})_{(0)}$, the 2-section $L[\alpha, \beta]$ is semisimple (not just T -semisimple), and \tilde{S} is the unique minimal ideal of $L[\alpha, \beta]$. On the other hand, applying Proposition 3.2 with $\mathfrak{t} = T \cap \ker \nu$ shows that the unique minimal ideal of $L[\alpha, \beta]$ is a simple Lie algebra (notice that $\mathfrak{c}_L(\mathfrak{t}) = L(\nu)$ is nilpotent by the Engel–Jacobson theorem). But then $m = 0$, a contradiction. This means that the case where $t_0 \in W(m; \underline{1})_{(0)}$ cannot occur.

Part B. Thus, we may assume that $t_0 \notin W(m; \underline{1})_{(0)}$. Because of [P-St 99, Thm. 3.2] it can be assumed further that $\overline{T} = F(h_0 \otimes 1) \oplus F(\text{Id}_S \otimes (1 + x_1)\partial_1)$. Then $\overline{H} \cap \tilde{S} = \mathfrak{c}_S(h_0) \otimes F[x_2, \dots, x_m]$. Since α and β are \mathbb{F}_p -independent, there exists $\lambda \in \mathbb{F}_p \alpha + \mathbb{F}_p \beta$ such that $\lambda(h_0 \otimes 1) = 0$ and $\lambda(\text{Id}_S \otimes (1 + x_1)\partial_1) = 1$. Note that

$$(4.5) \quad Fh_0 \otimes (1 + x_1)^i \subset \tilde{S}_{i\lambda} \subset \overline{(\text{rad } L(\lambda))}_{i\lambda} \quad \forall i \in \mathbb{F}_p^\times.$$

Hence $(\text{rad } L(\lambda))_{i\lambda}$ contains nonnilpotent elements of L_p for all $i \in \mathbb{F}_p^\times$. Lemma 4.4(b) yields $\lambda \notin \Omega$. Since $S \otimes F$ is \overline{T} -stable and not nilpotent, there is $\kappa \in \Gamma(L[\alpha, \beta], \overline{T})$ with $(S \otimes F)_\kappa \neq (0)$. As $\kappa(\text{Id}_S \otimes (1 + x_1)\partial_1) = 0$, it must be that $\kappa(h_0 \otimes 1) \neq 0$.

Claim 1. If $\gamma \in \Gamma(L[\alpha, \beta], \overline{T})$, then $\gamma \in \Omega \Leftrightarrow \gamma(h_0 \otimes 1) \neq 0$.

As $\alpha \in \Omega$ and $\lambda \notin \Omega$, one has $i\alpha + j\lambda \in \Omega$ for all $i \in \mathbb{F}_p^\times$ and $j \in \mathbb{F}_p$. So

$$i\alpha + j\lambda \in \Omega \Leftrightarrow i \neq 0 \Leftrightarrow (i\alpha + j\lambda)(h_0 \otimes 1) \neq 0 \quad \forall i, j \in \mathbb{F}_p.$$

Since α and λ are \mathbb{F}_p -independent, their \mathbb{F}_p -span contains $\Gamma(L[\alpha, \beta], \overline{T})$. ✓

It follows from Claim 1 and (4.5) that $\Gamma(L[\alpha, \beta], \overline{T}) \setminus \Omega = \mathbb{F}_p^\times \lambda$ and L_γ contains nonnilpotent elements of L_p for all $\gamma \in \Gamma(L[\alpha, \beta], \overline{T}) \setminus \Omega$. Thus, it remains to show that $m = 1$.

Claim 2. The subspace $\sum_{j=2}^m S \otimes x_j \mathcal{O}(m; \underline{1})$ is \overline{H} -invariant.

Note that $L[\alpha, \beta](\kappa) = \overline{H} + S \otimes F[x_2, \dots, x_m]$. In particular, κ is nonsolvable. Let $\psi: L[\alpha, \beta](\kappa) \rightarrow L[\kappa]$ denote the canonical homomorphism. By Theorem 3.3, the Lie algebra $L[\kappa]^{(1)}$ is simple. As the ideal $S \otimes F[x_2, \dots, x_m]$ is perfect, ψ maps it onto $L[\kappa]^{(1)}$. As a consequence, $S \otimes F[x_2, \dots, x_m]_{(1)} = \ker \psi \cap (S \otimes F[x_2, \dots, x_m])$, showing that $S \otimes F[x_2, \dots, x_m]_{(1)}$ is \overline{H} -invariant. ✓

Claim 3. $S \cong H(2; \underline{1})^{(2)}$ and $[D, [D, h]]$ acts nonnilpotently on \tilde{S} for some $D \in \overline{H}$ and $h \in \overline{H} \cap \tilde{S}$.

We have seen in the proof of Claim 2 that

$$L[\kappa] = \psi(L[\alpha, \beta](\kappa)) \cong L[\alpha, \beta](\kappa) / \text{rad}(L[\alpha, \beta](\kappa)) \cong S + H / (H \cap \text{rad } L(\kappa)).$$

Our choice of κ and Claim 1 imply that $\kappa \in \Omega$. So Theorem 3.3 implies that $L[\kappa] \cong H(2; \underline{1})^{(2)} \oplus F\bar{D}$ and there exists $\tilde{h} \in \sum_{i \in \mathbb{F}_p^\times} [L_{i\kappa}, L_{-i\kappa}]$ such that $[\bar{D}, [\bar{D}, \Psi_\kappa(\tilde{h})]]$ acts nonnilpotently on $L[\kappa]$. Pick $D \in \psi^{-1}(\bar{D}) \cap \overline{H}$ and set $h := \Psi_{\alpha, \beta}(\tilde{h})$. Standard toral rank considerations show that $\ker \psi$ acts nilpotently on $L[\alpha, \beta](\kappa)$ (see the proof of Claim 7 in Part A for a similar argument). In light of the preceding remark this implies that $\kappa([D, [D, h]]) \neq 0$. ✓

Claim 4. $m = 1$.

We first note that $L[\alpha, \beta] \subset \overline{L(\lambda)} + (\text{Der } S) \otimes \mathcal{O}(m; \underline{1})$. If all derivations from the set $\bigcup_{i \in \mathbb{F}_p^\times} \text{Id}_S \otimes \pi(\overline{L_{i\lambda}})$ preserve the ideal $I := \sum_{j=2}^m S \otimes x_j \mathcal{O}(m; \underline{1})$ of $(\text{Der } S) \otimes \mathcal{O}(m; \underline{1})$, then Claim 2 entails that I is a nilpotent \overline{T} -stable ideal of $L[\alpha, \beta]$. Since $L[\alpha, \beta]$ is T -semisimple, this would force $m = 1$.

So assume for a contradiction that there exists $E \in L_{k\lambda}$ for some $k \in \mathbb{F}_p^\times$ such that $\text{Id}_S \otimes \pi(\overline{E})$ does not preserve I . Since $\pi(\overline{E})$ is an eigenvector for $(1 + x_1)\partial_1$ with eigenvalue $k \neq 0$, it has the form

$$\pi(\overline{E}) = f_1(x_2, \dots, x_m)(1 + x_1)^{k+1}\partial_1 + \sum_{j=2}^m f_j(x_2, \dots, x_m)(1 + x_1)^k\partial_j$$

for some $f_1, \dots, f_m \in F[x_2, \dots, x_m]$. As $\pi(\overline{E})$ does not stabilize I , it must be that $f_{j_0}(0) \neq 0$ for some $j_0 \geq 2$. After renumbering we may assume that $j_0 = 2$. Since $\mathfrak{c}_S(h_0) \otimes (1 + x_1)^{p-k}x_2 \subset \overline{\tilde{S}_{-k\lambda}} \subset \overline{(\text{rad } L(\lambda))_{-k\lambda}}$, we have that

$$\begin{aligned} \mathfrak{c}_S(h_0) \otimes F &\subset [\overline{E}, \tilde{S}_{-k\lambda}] + \left(S \otimes F[x_1, \dots, x_m]_{(1)} \right) \cap \overline{H} \\ &= [\overline{E}, \tilde{S}_{-k\lambda}] + \mathfrak{c}_S(h_0) \otimes F[x_1, \dots, x_m]_{(1)}. \end{aligned}$$

From this it follows that

$$\overline{H} \cap \tilde{S} = \mathfrak{c}_S(h_0) \otimes F[x_1, \dots, x_m] \subset [\overline{L_{k\lambda}}, \overline{(\text{rad } L(\lambda))_{-k\lambda}}] + \mathfrak{c}_S(h_0) \otimes F[x_2, \dots, x_m]_{(1)}.$$

The subspace $I \cap \overline{H} = \mathfrak{c}_S(h_0) \otimes F[x_2, \dots, x_m]_{(1)}$ is \overline{H} -invariant by Claim 2 and acts nilpotently on $L[\alpha, \beta](\kappa)$. These observations in conjunction with Claim 3 imply that $(\text{ad } D)^2([\overline{L}_{k\lambda}, \overline{(\text{rad } L(\lambda))}_{-k\lambda}]) \subset \overline{H}^3 \cap [\overline{L}_{k\lambda}, \overline{(\text{rad } L(\lambda))}_{-k\lambda}]$ does not consist of nilpotent derivations of \tilde{S} . But then $\lambda(H) = 0$ by Lemma 4.4(a).

We now set $\mathfrak{t} := T \cap \ker \lambda$. Since $L(\lambda) = \mathfrak{c}_L(\mathfrak{t})$ is nilpotent by the Engel–Jacobson theorem, Proposition 3.2 says that $L(\alpha, \beta)/\text{rad } L(\alpha, \beta)$ has a unique minimal ideal, S' say, which is a simple Lie algebra. Then S' must be the image of $\tilde{S} = S \otimes \mathcal{O}(m; \underline{1})$ under the natural homomorphism $\phi: L[\alpha, \beta] \rightarrow L(\alpha, \beta)/\text{rad } L(\alpha, \beta)$. As a consequence, $\ker \phi \cap \tilde{S}$ coincides with the radical of \tilde{S} . As the latter equals $S \otimes \mathcal{O}(m; \underline{1})_{(1)}$, we derive that $S \otimes \mathcal{O}(m; \underline{1})_{(1)} = \ker \phi \cap \tilde{S}$ is an ideal of $L[\alpha, \beta]$. On the other hand, $\pi(\overline{E}) \notin W(m; \underline{1})_{(0)}$. This shows that our present assumption is false and $m = 1$. \checkmark

The proof of the proposition is now complete. \square

Corollary 4.6. *Let $\alpha \in \Omega$, $\beta \in \Gamma(L, T)$ and suppose $L[\alpha, \beta]$ is as in case 4) of Proposition 4.2. Then $\sum_{i \in \mathbb{F}_p^\times} (\text{rad } L(\gamma))_{i\gamma} \subset \text{rad}_T L[\alpha, \beta]$ for all $\gamma \in \Omega \cap (\mathbb{F}_p \alpha + \mathbb{F}_p \beta)$.*

Proof. Pick $\gamma \in \Omega \cap (\mathbb{F}_p \alpha + \mathbb{F}_p \beta)$ and view it as a \overline{T} -root of $L[\alpha, \beta]$. In the present case $L[\alpha, \beta](\gamma) = \overline{H} + \tilde{S}(\gamma)$ and $\tilde{S} = H(2; \underline{1})^{(2)} \otimes \mathcal{O}(1; \underline{1})$; see Proposition 4.5. Furthermore, in the notation of Proposition 4.5 we have that $\gamma = i\kappa + j\lambda$ for some $i \in \mathbb{F}_p^\times$ and $j \in \mathbb{F}_p$, where $\kappa, \lambda \in \overline{T}^*$ are such that $\kappa(h_0 \otimes 1) = r \in \mathbb{F}_p^\times$, $\kappa(\text{Id}_S \otimes (1 + x_1)\partial_1) = 0$, $\lambda(h_0 \otimes 1) = 0$ and $\lambda(\text{Id}_S \otimes (1 + x_1)\partial_1) = 1$. Let S_ℓ denote the ℓ -eigenspace of $\text{ad}_S h_0$. Then

$$\tilde{S}(\gamma) = \bigoplus_{k \in \mathbb{F}_p} S_{kir} \otimes (1 + x_1)^{kj} \cong \bigoplus_{k \in \mathbb{F}_p} S_{kir} = H(2; \underline{1})^{(2)}$$

as Lie algebras. Hence $\text{rad}(L[\alpha, \beta](\gamma)) = \text{rad}(\overline{H} + \tilde{S}(\gamma)) \subset \overline{H}$. The result follows. \square

We are now in a position to prove our first result on the global structure of L .

Theorem 4.7. *If $\alpha \in \Omega$, then α is Hamiltonian, $\dim L_\alpha = 5$, and $\text{rad } L(\alpha) \subset H$.*

Proof. For $\gamma \in \Gamma(L, T)$ put $R_\gamma := (\text{rad } L(\gamma))_\gamma$. Let $\mu \in \Omega$ be such that $\text{rad } L(\mu) \not\subset H$. By Theorem 3.3, the radical of $L(\mu)$ is T -stable. Hence there is $a \in \mathbb{F}_p^\times$ such that $(\text{rad } L(\mu))_{a\mu} \neq (0)$. Put $\nu := a\mu$ and note that $\nu \in \Omega$. For $k \in \mathbb{Z}_+$ define

$$I_0 := R_\nu, \quad I_k := \sum_{\gamma_1, \dots, \gamma_k} [L_{\gamma_1}, [\dots [L_{\gamma_k}, R_\nu] \dots]], \quad I := \sum_{k \geq 0} I_k.$$

Clearly, I is an ideal of L containing R_ν . We intend to show that $I \subsetneq L$. As a first step we are going to use induction on k to prove the following:

Claim. *If $\nu + \gamma_1 + \dots + \gamma_k \in \Omega$, then $[L_{\gamma_1}, [\dots [L_{\gamma_k}, R_\nu] \dots]] \subset R_{\nu + \gamma_1 + \dots + \gamma_k}$.*

The claim is obviously true for $k = 0$, and it also holds for $k = 1$ thanks to Corollaries 4.3 and 4.6. Suppose it is true for all $k < n$ and let $\gamma_1, \dots, \gamma_n \in \Gamma(L, T)$ be such that $\nu + \gamma_1 + \dots + \gamma_n \in \Omega$. If $\nu + \gamma_i \in \Omega$ or $\nu + \gamma_i \notin \Gamma(L, T)$ for some $i \leq n$, then applying Corollaries 4.3 and 4.6 gives

$$\begin{aligned} [L_{\gamma_1}, [\dots [L_{\gamma_n}, R_\nu] \dots]] &\subset [L_{\gamma_1}, [\dots [\widehat{L_{\gamma_i}} \dots [L_{\gamma_n}, [L_{\gamma_i}, R_\nu]] \dots] \dots]] + I_{n-1} \\ &\subset [L_{\gamma_1}, [\dots [\widehat{L_{\gamma_i}} \dots [L_{\gamma_n}, R_{\nu + \gamma_i}] \dots] \dots]] + I_{n-1}. \end{aligned}$$

In this case the claim holds by our induction hypothesis. So assume from now that $\nu + \gamma_i \in \Gamma(L, T) \setminus \Omega$ for all $i \leq n$. We may also assume that $\tilde{\nu} := \nu + \gamma_1 + \dots + \gamma_n$ is

not solvable, for otherwise we are done. According to Lemma 4.1 there is $\kappa \in \Gamma(L, T)$ such that $L[\tilde{\nu}, \kappa] \cong \mathcal{M}(1, 1)$. Moreover, it follows from [P 94, Lemmas 4.1 & 4.4] that the radical of every 1-section $L[\tilde{\nu}, \kappa](\delta)$ is contained in $\Psi_{\tilde{\nu}, \kappa}(T)$ and

$$(4.6) \quad (\mathbb{F}_p \tilde{\nu} + \mathbb{F}_p \kappa) \setminus \{0\} \subset \Omega.$$

Take an arbitrary $\kappa' \in (\mathbb{F}_p \tilde{\nu} + \mathbb{F}_p \kappa) \setminus \mathbb{F}_p \tilde{\nu}$. It follows from (4.6) that $\tilde{\nu} + \mathbb{F}_p \kappa' \subset \Gamma(L, T)$. Note that the rule

$$\gamma \asymp \gamma' \Leftrightarrow (\gamma - \gamma')|_{H^3} = 0$$

defines an equivalence relation on the set of all F -valued functions on H . Since $\gamma_i \asymp -\nu$ for all $i \leq n$, we have that $\tilde{\nu} \asymp (1-n)\nu$. If $\nu + \kappa' \asymp 0$, then $\tilde{\nu} + (1-n)\kappa' \notin \Omega$. As $\tilde{\nu} + (1-n)\kappa' \neq 0$ by our choice of κ' , this is not true; see (4.6). Thus, $\nu + \kappa' \not\asymp 0$, showing that $\nu + \kappa' \in \Omega$ whenever $\nu + \kappa' \in \Gamma(L, T)$. But then $[R_\nu, L_{\kappa'}] \subset R_{\nu+\kappa'}$ by Corollaries 4.3 and 4.6. As $\nu + \gamma_i \asymp 0$ and $\kappa' \in \Omega$ by (4.6), we also have that $\nu + (\gamma_i + \kappa') \in \Omega$ whenever $\nu + (\gamma_i + \kappa') \in \Gamma(L, T)$ for all $i \leq n$. So arguing as above one now obtains that $[[L_{\gamma_i}, L_{\kappa'}], R_\nu] \subset R_{\nu+\gamma_i+\kappa'}$. This implies that

$$[[L_{\gamma_1}, [\cdots [L_{\gamma_n}, R_\nu] \cdots]], L_{\kappa'}] \subset R_{\tilde{\nu}+\kappa'} \subset \text{rad } L(\tilde{\nu}, \kappa').$$

As $\mathcal{M}(1, 1)$ is a simple Lie algebra, Schue's lemma [St 04, Prop. 1.3.6(1)] yields

$$[\Psi_{\tilde{\nu}, \kappa'}([L_{\gamma_1}, [\cdots [L_{\gamma_n}, R_\nu] \cdots]]), \mathcal{M}(1, 1)] = (0),$$

forcing $[L_{\gamma_1}, [\cdots [L_{\gamma_n}, R_\nu] \cdots]] \subset (\text{rad } L(\tilde{\nu}, \kappa'))_{\tilde{\nu}} \subset (\text{rad } L(\tilde{\nu}))_{\tilde{\nu}} \subset R_{\tilde{\nu}}$. This completes the induction step.

As a consequence, $I_\gamma \subset R_\gamma$ for all $\gamma \in \Omega$. On the other hand, it follows from [P 94, Lemma 3.8] that Ω contains at least one Hamiltonian root, λ say. Then $I_\lambda \neq L_\lambda$, implying $I \neq L$. Then $I = (0)$, proving that $\text{rad } L(\mu) \subset H$ for all $\mu \in \Omega$. As a consequence, all roots in Ω are nonsolvable.

Now let $\alpha \in \Omega$. Because α is nonsolvable, it follows from Theorem 3.3 that α is Hamiltonian. Since $\text{rad } L(\alpha) \subset H$, this gives $\dim L_\alpha = 5$. \square

5. Further reductions

In this section we are going to prove that no root in $\Gamma(L, T)$ vanishes on H^3 . Theorem 4.7 will play a crucial role in our arguments.

Lemma 5.1. *If $\gamma \in \Gamma(L, T)$ does not vanish on H , then $\gamma \in \Omega$.*

Proof. Suppose there is $\beta \in \Gamma(L, T) \setminus \Omega$ such that $\beta(H) \neq 0$. By (4.1), there is $\alpha \in \Omega$ such that $\beta([L_\alpha, L_{-\alpha}]) \neq 0$. Then $[L_\beta, [L_\alpha, L_{-\alpha}]] = L_\beta$, implying that $\alpha + \beta \in \Gamma(L, T)$ or $-\alpha + \beta \in \Gamma(L, T)$. Since $\beta \notin \Omega$ by our assumption, we have that $\alpha + \beta \in \Omega$ or $-\alpha + \beta \in \Omega$. Theorem 4.7 then shows that $\{\alpha, \alpha + \beta\}$ or $\{\alpha, -\alpha + \beta\}$ consists of nonsolvable roots. Then $L[\alpha, \beta]$ cannot be of type 1) or 2) of Proposition 4.2.

Suppose $L[\alpha, \beta]$ is as in case 3) of Proposition 4.2 and set $\delta_1 := \alpha$, $\delta_2 := \alpha + \beta$ if $\alpha + \beta \in \Gamma(L, T)$ and $\delta_1 := \alpha$, $\delta_2 := \alpha - \beta$ if $-\alpha + \beta \in \Gamma(L, T)$. In either case, we can find elements $h_1, h_2 \in H^3$ such that $\delta_i(h_j) = \delta_{ij}$ for $i, j \in \{1, 2\}$. As a consequence, $\alpha(h_2) = 0$ and $\beta(h_2) \neq 0$. But then $\beta \in \Omega$, a contradiction.

Suppose $L[\alpha, \beta]$ is as in case 4) of Proposition 4.2. Then Proposition 4.5 applies. As $\alpha \in \Omega$, Proposition 4.5 says that $\alpha(h_0 \otimes 1) \neq 0$. This forces $\Psi_{\alpha, \beta}(L_{\pm\alpha}) \subset \tilde{S}$. Since

$\beta([L_\alpha, L_{-\alpha}]) \neq 0$, we now deduce that β does not vanish on $\Psi_{\alpha,\beta}(H) \cap \tilde{S}$. This forces $\beta(h_0 \otimes 1) \neq 0$. Applying Proposition 4.5 once again we obtain $\beta \in \Omega$, a contradiction.

Suppose $L[\alpha, \beta]$ is of type 5) of Proposition 4.2. Then $\tilde{S} = H(2; (2, 1))^{(2)}$ and $L[\alpha, \beta] \subset H(2; (2, 1))$. In this case $\Psi_{\alpha,\beta}(H)^3 \subset \tilde{S}$, and it follows from Lemma 2.5 and Demuškin's description of maximal tori in $H(2; \underline{1})^{(2)}$ that $\Psi_{\alpha,\beta}(H) \cap \tilde{S}$ is abelian and $\text{nil}(\Psi_{\alpha,\beta}(H) \cap \tilde{S})$ has codimension 1 in $\Psi_{\alpha,\beta}(H) \cap \tilde{S}$; see [St 04, Thm. 7.5.8] for instance. As $\alpha \in \Omega$, this means that $\Psi_{\alpha,\beta}(H) \cap \tilde{S} = \Psi_{\alpha,\beta}(H)^3 + \text{nil}(\Psi_{\alpha,\beta}(H) \cap \tilde{S})$. As a consequence, $\gamma \in \Gamma(L[\alpha, \beta], \Psi_{\alpha,\beta}(T))$ is in Ω if and only if $\gamma(\Psi_{\alpha,\beta}(H) \cap \tilde{S}) \neq 0$. As $\alpha \in \Omega$, Theorem 4.7 implies that α does not vanish on $[\Psi_{\alpha,\beta}(L_\alpha), \Psi_{\alpha,\beta}(L_{-\alpha})]$. As $\Psi_{\alpha,\beta}(L_{\pm\alpha}) \subset \tilde{S}$, this shows that

$$\Psi_{\alpha,\beta}(H) \cap \tilde{S} = [\Psi_{\alpha,\beta}(L_\alpha), \Psi_{\alpha,\beta}(L_{-\alpha})] + \text{nil}(\Psi_{\alpha,\beta}(H) \cap \tilde{S}).$$

But then $\beta(\Psi_{\alpha,\beta}(H) \cap \tilde{S}) \neq 0$ by our choice of β , implying that $\beta \in \Omega$. Since this contradicts our choice of β , we derive that $L[\alpha, \beta]$ cannot be of type 5).

If $L[\alpha, \beta]$ is as in case 6) of Proposition 4.2, then $(\mathbb{F}_p\alpha + \mathbb{F}_p\beta) \setminus \{0\} \subset \Omega$ by [P 94, Lemmas 4.1 & 4.4]. So this case cannot occur either, and our proof is complete. \square

Proposition 5.2. *If $\mu \in \Gamma(L, T)$ vanishes on H , then L_μ consists of p -nilpotent elements of L_p .*

Proof. Suppose for a contradiction that there is $\mu \in \Gamma(L, T)$ with $\mu(H) = 0$ such that $\alpha(L_\mu^{[p]}) \neq 0$ for some $\alpha \in \Gamma(L, T)$. It follows from (4.1) that every root is the sum of two roots in Ω . Therefore, we may assume that $\alpha \in \Omega$. Since α is nonsolvable by Theorem 4.7, there exists $\beta \in \Omega$ such that $L[\alpha, \beta] \cong \mathcal{M}(1, 1)$ and $\alpha([L_{i\alpha}, L_{-i\alpha}], [L_\beta, L_{-\beta}]) \neq 0$ for some $i \in \mathbb{F}_p^\times$; see Lemma 4.1. Lemma 5.1 shows that $\beta \in \Omega$.

We now consider the T -semisimple 3-section $L[\alpha, \beta, \mu]$. Set $\overline{T} := \Psi_{\alpha,\beta,\mu}(T)$, $\overline{H} := \Psi_{\alpha,\beta,\mu}(H)$ and $\tilde{S} := \tilde{S}(\alpha, \beta, \mu)$. Given a Lie subalgebra M of $L[\alpha, \beta, \mu]$ we denote by $M_{[p]}$ the p -envelope of M in $\text{Der } \tilde{S}$. Note that the restricted Lie algebra $\overline{T} + L[\alpha, \beta, \mu]_{[p]} \subset \text{Der } \tilde{S}$ is centerless. As T is a torus of maximal dimension in $T + L(\alpha, \beta, \mu)_p$, it follows from [St 04, Thm. 1.2.8(4a)] that \overline{T} is a torus of maximal dimension in $\overline{T} + L[\alpha, \beta, \mu]_{[p]}$. Let J be a minimal T -invariant ideal of $L[\alpha, \beta, \mu]$. Then $TR(J) \leq TR(L[\alpha, \beta, \mu]) \leq 3$; see [St 04, Thms 1.2.7(1) & 1.3.11(3)].

(a) Suppose $TR(J) = 3$. Then it follows from [St 04, Thm. 1.2.9(3)] that the restricted Lie algebra $(\overline{T} + L[\alpha, \beta, \mu]_{[p]})/J_{[p]}$ is p -nilpotent. From this it is immediate that $\overline{T} \subset J_{[p]}$, $J = \tilde{S}$ and $L[\alpha, \beta, \mu] = \overline{H} + \tilde{S}$. By Block's theorem, $\tilde{S} = S \otimes \mathcal{O}(m; \underline{1})$, where S is a simple Lie algebra and $m \in \mathbb{Z}_+$. Let π denote the canonical projection

$$\text{Der}(S \otimes \mathcal{O}(m; \underline{1})) = (\text{Der } S) \otimes \mathcal{O}(m; \underline{1}) \rtimes \text{Id}_S \otimes W(m; \underline{1}) \twoheadrightarrow W(m; \underline{1}).$$

In the present situation [P-St 99, Thm. 2.6] implies that the torus \overline{T} is conjugate under $\text{Aut}(S \otimes \mathcal{O}(m; \underline{1}))$ to $T_0 \otimes F$ for some torus T_0 in S_p . Hence we can choose $\Psi_{\alpha,\beta,\mu}$ such that $\overline{T} = T_0 \otimes F$. Then $L[\alpha, \beta, \mu](\alpha) = \overline{H} + S(\alpha) \otimes \mathcal{O}(m; \underline{1})$. Since α is nonsolvable, there is a surjective homomorphism $\psi: L[\alpha, \beta, \mu](\alpha) \twoheadrightarrow L[\alpha] \neq (0)$. By Theorem 3.3, $(\text{im } \psi)^{(1)}$ is a simple Lie algebra and the unique minimal ideal of

im ψ . Since T_0 is a torus of maximal dimension in S_p , Theorem 3.3 also applies to the 1-section $S[\alpha]$. So it must be that $(\text{im } \psi)^{(1)} \cong S[\alpha]^{(1)}$. As a consequence,

$$\tilde{S}(\alpha)^{(1)} \cap \ker \psi = (\text{rad } S(\alpha) \cap S(\alpha)^{(1)}) \otimes F + S(\alpha)^{(1)} \otimes \mathcal{O}(m; \underline{1})_{(1)}$$

is \overline{H} -invariant. As $S(\alpha)$ is not solvable, it follows that $\pi(\overline{H}) \subset W(m; \underline{1})_{(0)}$. But then $S \otimes \mathcal{O}(m; \underline{1})_{(1)}$ is an ideal of $L[\alpha, \beta, \mu]$. As $L[\alpha, \beta, \mu]$ is T -semisimple and $T = T_0 \otimes F$, we now obtain that $m = 0$ and $L[\alpha, \beta, \mu] = \overline{H} + \tilde{S}$.

As a consequence, $\Psi_{\alpha, \beta, \mu}(L_\gamma) \subset \tilde{S}$ for all $\gamma \in \Gamma(L[\alpha, \beta, \mu], \overline{T})$. This implies that $L[\alpha, \beta] \cong \mathcal{M}(1, 1)$ is a homomorphic image of the 2-section $\tilde{S}(\alpha, \beta)$, showing that $\overline{H} \cap \tilde{S}$ is a nontriangulable subalgebra of \tilde{S} . We now set $\mathfrak{t} := \Psi_{\alpha, \beta, \mu}(T \cap \ker \mu)$ and $\mathfrak{h} := \tilde{S}(\mu)$. Then \tilde{S} is simple, \mathfrak{t} is a torus of dimension at most 2 in \tilde{S}_p , and $\overline{H} \cap \tilde{S} \subset \mathfrak{h}$. This inclusion in conjunction with our assumption on μ and the Engel–Jacobson theorem shows that \mathfrak{h} is a nontriangulable nilpotent subalgebra of \tilde{S} . But then [P 94, Thm. 1(ii)] yields $\tilde{S} \cong \mathcal{M}(1, 1)$. As $TR(\mathcal{M}(1, 1)) = 2$ by [P 94, Lemma 4.3], we reach a contradiction thereby establishing that $TR(J) \leq 2$.

(b) We now put $T' := \overline{T} \cap J_{[p]}$ and observe that

$$\dim T' \geq TR(J_{[p]}, \overline{T} + L[\alpha, \beta, \mu]_{[p]}) = TR(J_{[p]}) \neq 0;$$

see [St 04, Thms 1.2.9 & 1.2.8(2)] (one should also keep in mind that $\overline{T} + L[\alpha, \beta, \mu]_{[p]}$ is centerless).

Suppose $\mu(T') \neq 0$. Then $\Psi_{\alpha, \beta, \mu}(L_{i\mu}) \subset J$ for all $i \in \mathbb{F}_p^\times$ and hence $\Psi_{\alpha, \beta, \mu}(L_\alpha) \subset J$ by our choice of α . Since $L[\alpha, \beta] \cong \mathcal{M}(1, 1)$ is simple, it follows that $\Psi_{\alpha, \beta, \mu}(L_{i\alpha+j\beta}) \subset J$ for all nonzero $(i, j) \in \mathbb{F}_p^2$. As a consequence, the p -envelope of $\overline{H} \cap J$ in $J_{[p]}$ contains a torus of dimension at least 2. This torus must be smaller than T' , because μ vanishes on H . But then $TR(J) > 2$ which is not true.

Thus, $\mu(T') = 0$. Then $\alpha(T') \neq 0$ or $\beta(T') \neq 0$. Relying on the simplicity of $L[\alpha, \beta] \cong \mathcal{M}(1, 1)$ and arguing as before, we derive that $J(\alpha, \beta)/\text{rad } J(\alpha, \beta) \cong \mathcal{M}(1, 1)$. As $\mu(T') = 0$, it follows that $\dim T' = TR(J) = 2$. By Block's theorem, $J = J' \otimes \mathcal{O}(k; \underline{1})$ for some simple Lie algebra J' and some $k \in \mathbb{Z}_+$. The above shows that $TR(J') = 2$. The natural homomorphism $J \rightarrow J/J' \otimes \mathcal{O}(k; \underline{1})_{(1)} \cong J'$ maps $J(\alpha, \beta)$ onto a subalgebra \mathfrak{g} of J' such that $\mathfrak{g}/\text{rad } \mathfrak{g} \cong \mathcal{M}(1, 1)$. As $TR(J') = 2$, this implies that J_p contains a nonstandard 2-dimensional torus. Applying [P 94, Thm. 1(ii)] now yields $J' \cong \mathcal{M}(1, 1)$. Since this holds for every minimal \overline{T} -invariant ideal of $L[\alpha, \beta, \mu]$ and $TR(L[\alpha, \beta, \mu]) \leq 3$, we may conclude at this point that the T -socle $\tilde{S} = \tilde{S}(\alpha, \beta, \mu) = S \otimes \mathcal{O}(m; \underline{1})$ is the unique minimal ideal of $L[\alpha, \beta, \mu]$.

Recall that all derivations of $S = \mathcal{M}(1, 1)$ are inner; see [St 04, Thm. 7.1.4] for instance. In this situation [P-St 99, Thm. 3.2] says that $\Psi_{\alpha, \beta, \mu}$ can be chosen such that $\overline{T} = (T_0 \otimes 1) + F(\text{Id}_S \otimes t_0)$, where T_0 is a 2-dimensional torus in $S_p = S$ and $t_0 \in W(m; \underline{1})$. Furthermore, $L[\alpha, \beta, \mu] = \mathcal{M}(1, 1) \otimes \mathcal{O}(m; \underline{1}) \rtimes \text{Id}_S \otimes \mathfrak{d}$ for some Lie subalgebra \mathfrak{d} of $W(m; \underline{1})$. Note that $T' = \overline{T} \cap \tilde{S} = T_0 \otimes 1$. Using the simplicity of $L[\alpha, \beta]$ and arguing as before, we observe that $\Psi_{\alpha, \beta, \mu}(L_{i\alpha+j\beta}) \subset \tilde{S}$ for all nonzero $(i, j) \in \mathbb{F}_p^\times$. By the choice of β , we then have $\alpha([\tilde{S}_{i\alpha}, \tilde{S}_{-i\alpha}], [\tilde{S}_\beta, \tilde{S}_{-\beta}]) \neq 0$ for some $i \in \mathbb{F}_p^\times$. This means that T_0 is a nonstandard torus in $S = \mathcal{M}(1, 1)$.

If $t_0 \notin W(m; \underline{1})_{(0)}$, then we may assume further that $t_0 = (1 + x_1)\partial_1$; see [P-St 99, Thm. 3.2]. Choose $h, h' \in \mathfrak{c}_S(T_0)$ such that $[h, h']$ acts nonnilpotently on S . Recall that $\mu(T_0 \otimes F) = 0$. Then $\mu(\text{Id}_S \otimes t_0) \neq 0$ and hence there exists $r \in \mathbb{F}_p^\times$ such that $h \otimes (1 + x_1) \in \tilde{S}_{r\mu}$ and $h' \otimes (1 + x_1)^{p-1} \in \tilde{S}_{-r\mu}$. Clearly, the element

$$[h \otimes (1 + x_1), h' \otimes (1 + x_1)^{p-1}] \in [\tilde{S}_{r\mu}, \tilde{S}_{-r\mu}]$$

acts nonnilpotently on \tilde{S} .

Suppose $t_0 \in W(m; \underline{1})_{(0)}$. Since \tilde{S} is $(\text{Id}_S \otimes (Ft_0 + \mathfrak{d}))$ -simple, there is $r \in \mathbb{F}_p$ such that $\mathfrak{d}_{r\mu} \not\subset W(m; \underline{1})_{(0)}$ (here $\mathfrak{d}_0 = \pi(\overline{H})$ is the centraliser of t_0 in \mathfrak{d}). On the other hand, looking at the 1-section $L[\alpha, \beta, \mu](\alpha) = \overline{H} + S(\alpha) \otimes \mathcal{O}(m; \underline{1})$ and applying Theorem 3.3 to $L[\alpha] \neq (0)$ one observes that $\pi(\overline{H}) \subset W(m; \underline{1})_{(0)}$ (see part (a) for a similar argument). So it must be that $t_0 \neq 0$ and $r \in \mathbb{F}_p^\times$.

Let $E \in L_{r\mu}$ be such that $\pi(\Psi_{\alpha, \beta, \mu}(E)) \equiv \sum_{j=1}^m a_j \partial_j \pmod{W(m; \underline{1})_{(0)}}$, where not all a_j are zero. We may assume after renumbering and rescaling that $a_1 = 1$. In the present situation [P-St 99, Thm. 3.2] says that $\Psi_{\alpha, \beta, \gamma}$ can be chosen such that $t_0 = \sum_{j=1}^m s_j x_j \partial_j$ for some $s_j \in \mathbb{F}_p$. As $[t_0, \pi(\Psi_{\alpha, \beta, \mu}(E))]$ is a *nonzero* multiple of $\pi(\Psi_{\alpha, \beta, \mu}(E))$, it must be that $s_1 \neq 0$. Therefore, $\mathfrak{c}_S(T_0) \otimes x_1 \subset \tilde{S}_{-r\mu}$, implying that $[\Psi_{\alpha, \beta, \mu}(L_{r\mu}), \tilde{S}_{-r\mu}]$ contains nonnilpotent elements of \tilde{S} .

(c) We have thus shown that there is $r \in \mathbb{F}_p^\times$ such that $[L_{r\mu}, L_{-r\mu}]$ contains nonnilpotent elements of L_p . Therefore, the set

$$\Omega_1 := \{\gamma \in \Gamma(L, T) \mid \gamma([L_{r\mu}, L_{-r\mu}]) \neq 0\}.$$

is nonempty. By Lemma 5.1, we have the inclusion $\Omega_1 \subset \Omega$. Also, $\mu \notin \Omega_1$, because $\mu(H) = 0$. Since $\mu \neq 0$, there is $\gamma \in \Gamma(L, T)$ such that $\mu(L_\gamma^{[p]}) \neq 0$.

Suppose $\gamma \in \Omega$. Since $\mu(L_\gamma^{[p]}) \neq 0$, all elements from $\mu + \mathbb{F}_p \gamma$ are in $\Gamma(L, T)$. Since $\mu(H) = 0$, we then have $\mu + \mathbb{F}_p^\times \gamma \subset \Omega$. Since all roots in Ω are nonsolvable by Theorem 4.7, the T -semisimple 2-section $L[\gamma, \mu]$ cannot be as in cases 1), 2) or 3) of Proposition 4.2. If $L[\gamma, \mu]$ is of type 4), then Proposition 4.5 implies that $\Psi_{\gamma, \mu}(L_\gamma) \subset \tilde{S}$. As $\mu(L_\gamma^{[p]}) \neq 0$, this forces $\Psi_{\gamma, \mu}(L_{i\mu}) \subset \tilde{S}$ for all $i \in \mathbb{F}_p^\times$. Since μ vanishes on H , it follows from the description of $\Psi_{\gamma, \mu}(T)$ given in Proposition 4.5 that

$$\sum_{i \in \mathbb{F}_p^\times} \Psi_{\gamma, \mu}(L_{i\mu}) \subset \mathfrak{c}_{H(2; \underline{1})^{(2)}}(h_0) \otimes \mathcal{O}(1; \underline{1}).$$

As the subalgebra on the right is abelian and $\Psi_{\gamma, \mu}(L_{i\gamma}) \neq (0)$ for all $i \in \mathbb{F}_p^\times$, this contradicts our choice of μ . So $L[\gamma, \mu]$ is not of that type. If $L[\gamma, \mu]$ is as in cases 5) or 6) of Proposition 4.2, then Corollary 4.3 shows that no root in $\Gamma(L[\gamma, \mu], \Psi_{\gamma, \mu}(T)) = (\mathbb{F}_p \gamma \oplus \mathbb{F}_p \mu) \setminus \{0\}$ vanishes on $\Psi_{\gamma, \mu}(H)$. As $\mu(H) = 0$, this is false.

Thus, $\gamma \notin \Omega$. Schue's lemma [St 04, Prop. 1.3.6(1)] yields $L_\gamma = \sum_{\delta \in \Omega_1} [L_\delta, L_{\gamma-\delta}]$. If $x_1 \dots, x_d \in L_\gamma$, then

$$\left(\sum_{j=1}^d x_j\right)^{[p]} \equiv \sum_{j=1}^d x_j^{[p]} \pmod{H},$$

by Jacobson's formula. Note that the set $H \cup \left(\bigcup_{\delta \in \Omega_1, k \geq 1} [L_\delta, L_{-\delta}]^{[p]^k}\right)$ is weakly closed. Since μ vanishes on H , the Engel–Jacobson theorem implies that there is $\kappa \in \Omega_1$ such that $\mu([L_\kappa, L_{\gamma-\kappa}]^{[p]}) \neq 0$. Note that κ and $\gamma - \kappa$ are both in Ω , hence

$\Psi_{\gamma,\kappa,\mu}(L_\kappa) \neq (0)$ and $\Psi_{\gamma,\kappa,\mu}(L_{\gamma-\kappa}) \neq (0)$ by Theorem 4.7. Let $\tilde{S} = \tilde{S}(\gamma, \kappa, \mu)$ and let J be any minimal ideal of $L[\gamma, \kappa, \mu]$. Put $T_1 := \Psi_{\gamma,\kappa,\mu}(T) \cap J_{[p]}$, where $J_{[p]}$ is the p -envelope of J in $\text{Der } \tilde{S}$. Since $J_{[p]}$ is centerless, it follows from [St 04, Thm. 1.2.8(a)] that T_1 is a torus of maximal dimension in $J_{[p]}$.

Suppose $\mu(T_1) = 0$. Then either $\kappa(T_1) \neq 0$ or $(\gamma - \kappa)(T_1) \neq 0$, for $T_1 \neq (0)$. In any event, $\Psi_{\gamma,\kappa,\mu}([L_\kappa, L_{\gamma-\kappa}]) \subset J$ and therefore $\mu(J_{[p]}) \neq 0$. But then $\mu(T_1) \neq 0$, a contradiction. Thus, $\mu(T_1) \neq 0$, forcing $\sum_{i \in \mathbb{F}_p^\times} \Psi_{\gamma,\kappa,\mu}(L_{i\mu}) \subset J$. As $\kappa \in \Omega_1$, this yields $\sum_{i \in \mathbb{F}_p^\times} \Psi_{\gamma,\kappa,\mu}(L_{i\kappa}) \subset J$. As a result, the nilpotent subalgebra $J(\mu)$ acts nontriangulably on J . As $\kappa([L_{r\mu}, L_{-r\mu}]) \neq 0$ and $\Psi_{\gamma,\kappa,\mu}([L_\kappa, L_{\gamma-\kappa}]^{[p]}) \subset J_{[p]}$, we have that $TR(J) = \dim T_1 \geq 2$ (one should keep in mind that μ vanishes on H but not on $[L_\kappa, L_{\gamma-\kappa}]^{[p]}$).

Since $\kappa \in \Omega$, we can now argue as in part (a) of this proof to deduce that $TR(J) \leq 2$. As a result, $TR(J) = 2$ for any minimal ideal J of $L[\gamma, \kappa, \mu]$. As $TR(L[\gamma, \kappa, \mu]) \leq 3$, this shows that $\tilde{S} = S \otimes \mathcal{O}(m; \underline{1})$ is the unique minimal ideal of $L[\gamma, \kappa, \mu]$ and $TR(\tilde{S}) = TR(S) = 2$. According to [P-St 99, Thm. 2.6], we can choose $\Psi_{\gamma,\kappa,\mu}$ such that

$$\Psi_{\gamma,\kappa,\mu}(T) = (T'_0 \otimes 1) + F(d \otimes 1 + \text{Id}_S \otimes t_0), \quad T'_0 \subset S_p, \quad d \in \text{Der } S, \quad t_0 \in W(m; \underline{1}).$$

Moreover, if d is an inner derivation of S , then we can assume further that $d = 0$. Since $T_1 = T'_0 \otimes 1$, we get $\dim T'_0 = 2$. Set $\mathfrak{t} := T'_0 + Fd$, a torus in $\text{Der } S$. The subalgebra $S \otimes F$ of \tilde{S} is invariant under the action of $\Psi_{\gamma,\kappa,\mu}(T)$. Given $\delta \in \Gamma((S \otimes F), \Psi_{\gamma,\kappa,\mu}(T))$ we denote by $\bar{\delta}$ the unique \mathfrak{t} -root in $\Gamma(S, \mathfrak{t})$ for which $S_{\bar{\delta}} \otimes F = (S \otimes F)_\delta$.

(d) Suppose $t_0 \in W(m; \underline{1})_{(0)}$. Because \tilde{S} and $S \otimes \mathcal{O}(m; \underline{1})_{(1)}$ are both T -invariant, T acts on $S \cong \tilde{S}/(S \otimes \mathcal{O}(m; \underline{1})_{(1)})$ as the torus $\mathfrak{t} \subset \text{Der } S$. Since $\tilde{S}_\kappa \neq (0)$ and $\kappa \in \Omega_1$, we also have that $\Psi_{\gamma,\kappa,\mu}(L_{\pm r\mu}) \neq (0)$. We mentioned above that $\Psi_{\gamma,\kappa,\mu}(L_{\pm r\mu}) \subset \tilde{S}$. Define $\mathfrak{t}_0 := \mathfrak{t} \cap \ker \bar{\mu}$. Then $\dim \mathfrak{t}_0 \leq 2$ and $\mathfrak{c}_S(\mathfrak{t}_0) = S(\bar{\mu})$. Because $S_p \otimes \mathcal{O}(m; \underline{1})_{(1)}$ is p -nilpotent and $\tilde{S}(\mu)$ acts nontriangulably on \tilde{S} by our discussion in part (c), the subalgebra $S(\bar{\mu})$ is nilpotent and acts nontriangulably on S . Applying [P 94, Thm. 2(ii)] now yields $S \cong \mathcal{M}(1, 1)$. But then all derivations of S are inner; see [St 04, Thm. 7.1.4] for example. Then $d = 0$ and \mathfrak{t} is a torus of maximal dimension in S_p . It follows that $S(\bar{\mu}) = \mathfrak{c}_S(\mathfrak{t}_0)$ is a Cartan subalgebra of toral rank 1 in S . Since such Cartan subalgebras are triangulable by [P 94, Thm. 2], our assumption on t_0 is false.

Thus, $t_0 \notin W(m; \underline{1})_{(0)}$. Recall that μ and κ are both nonzero on $T_1 = T'_0 \otimes 1$. Since μ vanishes on H and the nonsolvable root κ does not vanish on $\Psi_{\gamma,\kappa,\mu}([L_{i\kappa}, L_{-i\kappa}]) \subset \Psi_{\gamma,\kappa,\mu}(H) \cap \tilde{S}$ for some $i \in \mathbb{F}_p^\times$, the roots μ and κ are linearly independent on T_1 . Hence

$$\Psi_{\gamma,\kappa,\mu}(T) = T_1 \oplus (\Psi_{\gamma,\kappa,\mu}(T) \cap \ker \mu \cap \ker \kappa),$$

implying that $\pi(\Psi_{\gamma,\kappa,\mu}(T) \cap \ker \mu \cap \ker \kappa) \not\subset W(m; \underline{1})_{(0)}$. In that case [P-St 99, Thm. 2.6] says that $\Psi_{\gamma,\kappa,\mu}$ can be selected such that $d = 0$, $t_0 = (1 + x_1)\partial_1$, and $\Psi_{\gamma,\kappa,\mu}(T) \cap \ker \mu \cap \ker \kappa = F(\text{Id}_S \otimes t_0)$.

Then $\tilde{S}(\kappa, \mu) = S \otimes F[x_2, \dots, x_m]$ and the evaluation map $\text{ev}: \tilde{S}(\kappa, \mu) \rightarrow S$, taking $s \otimes f \in S \otimes F[x_2, \dots, x_m]$ to $f(0)s \in S$, is T -equivariant. As before, $S(\bar{\mu})$ acts nontriangulably on S . Since in the present case \mathfrak{t} is a torus of maximal dimension in

S_p , its 1-section $S(\bar{\mu})$ has toral rank 1 in S . Since such a Cartan subalgebra must act triangulably on S by [P 94, Thm. 2], we reach a contradiction, thereby proving the proposition. \square

Corollary 5.3. *The following are true:*

- (i) $\Gamma(L, T) = \Omega$.
- (ii) *If $\alpha, \beta \in \Gamma(L, T)$, then $L[\alpha, \beta]$ is not as in case 4) of Proposition 4.2.*

Proof. (1) Suppose $\Gamma(L, T) \neq \Omega$ and let $\lambda \in \Gamma(L, T) \setminus \Omega$. Take any $\alpha \in \Omega$ and consider the T -semisimple 2-section $L[\alpha, \lambda]$. By Theorem 4.7, $L(\alpha)$ is not solvable, hence $L[\alpha, \lambda]$ is not as in case 1) of Proposition 4.2. Because of Lemma 5.1 we have $\lambda(H) = 0$, hence $L(\lambda)$ is solvable. If $L[\alpha, \lambda]$ is as in cases 2), 3), 5) or 6) of Proposition 4.2, then $L_\lambda \subset \text{rad}_T L(\alpha, \lambda)$ by Corollary 4.3, hence

$$[L_\alpha, L_\lambda] \subset (\text{rad}_T L(\alpha, \lambda)) \cap L_{\alpha+\lambda} \subset (\text{rad}_T L(\alpha + \lambda))_{\alpha+\lambda} = (0)$$

by Theorem 4.7 (because $\alpha + \lambda \in \Omega$). If $L[\alpha, \lambda]$ is as in case 4) of Proposition 4.2, then it follows from Proposition 4.5 that L_λ contains nonnilpotent elements of L_p . Since this contradicts Proposition 5.2, we see that $L[\alpha, \lambda]$ is not of that type. As a consequence, $[L_\alpha, L_\lambda] = 0$ for all $\alpha \in \Omega$. But then (4.1) yields that L_λ is contained in the center of L . This contradiction proves the first statement.

(2) If $L[\alpha, \beta]$ is as in case 4) of Proposition 4.2, then Proposition 4.5 implies that one of the roots in $\Gamma(L, T) \cap (\mathbb{F}_p\alpha + \mathbb{F}_p\beta)$ is not contained in Ω . Since this is impossible by part (1), our proof is complete. \square

Corollary 5.4. *For every $\alpha \in \Gamma(L, T)$ the radical of $L(\alpha)$ lies in the center of H .*

Proof. Recall that $\text{rad } L(\alpha) \subset H$ by Theorem 4.7 and Corollary 5.3. Set

$$\Omega_2 := \{\gamma \in \Gamma(L, T) \mid \gamma([H, \text{rad } L(\alpha)]) \neq 0\}.$$

Suppose $\Omega_2 \neq \emptyset$ and let $\beta \in \Omega_2$. Since $\alpha, \beta \in \Omega$ by Corollary 5.3, Proposition 4.2 applies to $L[\alpha, \beta]$. Since α vanishes on $[H, \text{rad } L(\alpha)]$, the roots α and β are \mathbb{F}_p -independent. As α and β are both nonsolvable by Theorem 4.7, $L[\alpha, \beta]$ cannot be as in cases 1) or 2) of Proposition 4.2. It cannot be governed by cases 5) or 6) either, because in case 5) the radical of $L[\alpha, \beta](\alpha)$ is trivial by Proposition 2.5(2) and in case 6) the radical of $L[\alpha, \beta](\alpha)$ is contained in $\Psi_{\alpha, \beta}(T)$; see [P 94, Lemmas 4.1 & 4.4].

Thus, $L[\alpha, \beta]$ is as in case 3) of Proposition 4.2. But then $L[\alpha, \beta] = L[\alpha, \beta](\alpha) + L[\alpha, \beta](\beta)$ and $[L_\alpha, L_\beta] \subset \text{rad}_T L(\alpha, \beta)$. Since $(\alpha + \beta)([H, \text{rad } L(\alpha)]) \neq 0$, and $L(\alpha + \beta)$ is solvable, it must be that $\alpha + \beta \notin \Gamma(L, T)$. We now derive that $[L_\alpha, L_\beta] = (0)$ for all $\beta \in \Omega_2$. In view of Schue's lemma [St 04, Prop. 1.3.6(1)], this means that L_α lies in the center of L .

This contradiction shows that $\Omega_2 = \emptyset$. Hence the ideal $H_\alpha := [H, \text{rad } L(\alpha)]$ of H consists of p -nilpotent elements of L_p . Now let β be any root in $\Gamma(L, T)$. Since $H_\alpha \subset H^{(1)}$, it follows from Theorem 3.3 and (the proof of) Lemma 3.4 that $\Psi_\beta(H_\alpha) = (0)$. Then $[H_\alpha, L(\beta)] \subset \text{rad } L(\beta)$, forcing $[H_\alpha, L_\beta] = (0)$; see Theorem 4.7. As a result, $[H_\alpha, L] = (0)$, and hence $H_\alpha = (0)$ by the simplicity of L . This proves the corollary. \square

We are finally in a position to describe the 2-sections of L with respect to T . Let $\mathfrak{z}(H)$ denote the center of $H = \mathfrak{c}_L(T)$.

Theorem 5.5. *The following are true:*

- (i) $H^4 = (0)$ and $H^{[p]} \subset T$.
- (ii) $\dim H^2 = 3$ and $\dim H^3 = 2$.
- (iii) $H^3 \subset T$ and $\dim H/\mathfrak{z}(H) = 3$.
- (iv) $\mathfrak{z}(H) = H \cap T$.

Proof. (a) Let $\alpha \in \Gamma(L, T)$. Then $\alpha \in \Omega$ by Corollary 5.3(i). It is immediate from Theorem 3.3 that $H^4 \subset \text{rad } L(\alpha)$. Then $[H^4, L_\alpha] \subset (\text{rad } L(\alpha))_\alpha = (0)$ by Theorem 4.7. Since this holds for every root α and L is simple, we derive $H^4 = (0)$.

Let $\mathcal{N}(H_p)$ denote the set of all p -nilpotent elements of H_p . Since $\dim L_\gamma = 5$ for all $\gamma \in \Gamma(L, T)$ any p -nilpotent element $x \in \mathcal{N}(H_p)$ has the property that $(\text{ad } x)^5(\sum_{\gamma \in \Gamma(L, T)} L_\gamma) = 0$. Then $x^{[p]} = 0$ by the simplicity of L . The Jordan–Chevalley decomposition in H_p now yields $(H_p)^{[p]} \subset T$, forcing $H^{[p]} \subset T$. As a result, statement (i) follows, and we also deduce that $\mathcal{N}(H_p) = \{x \in H_p \mid x^{[p]} = 0\}$ and $H_p \subset H + T$.

Since $H^4 = (0)$ and $[T, H] = 0$, Jacobson’s formula implies that $(x+y)^{[5]} = x^{[5]} + y^{[5]}$ for all $x, y \in H_p$. Therefore, $\mathcal{N}(H_p)$ is a subspace of H . By the Jordan–Chevalley decomposition in H_p , we also get $H_p \subset \mathcal{N}(H_p) \oplus T$.

(b) Since $\Gamma(L, T) = \Omega$, it follows from Theorem 3.3 and (the proof of) Lemma 3.4 that $H^2 + \text{rad } L(\alpha)$ has codimension 2 in H for every $\alpha \in \Gamma(L, T)$. Since $\text{rad } L(\alpha) \subset \mathfrak{z}(H)$ by Corollary 5.4, there exist $x, y \in H$ such that $H = Fx + Fy + H^2 + \mathfrak{z}(H)$. As a consequence, $H^2 = F[x, y] + H^3$ and $H^3 = F[x, [x, y]] + F[y, [y, x]] + H^4$. As $H^4 = (0)$, this gives $\dim H^3 \leq 2$ and $\dim H^2 = 1 + \dim H^3$.

Let $\alpha, \beta \in \Gamma(L, T)$ be such that $L[\alpha, \beta] \cong \mathcal{M}(1, 1)$ (such a pair of roots exists by [P 94, Thm 1(ii)]). It is immediate from [P 94, Lemmas 4.1 & 4.4] that $\dim \Psi_{\alpha, \beta}(H^3) = 2$. Hence $\dim H^3 \geq 2$. In conjunction with the above remarks, this gives $\dim H^3 = 2$ and $\dim H^2 = 3$. Statement (ii) follows.

(c) Since $H^4 = (0)$, we have that $H^3 \subset \mathfrak{z}(H)$. If the nilpotent Lie algebra $H/\mathfrak{z}(H)$ has codimension < 3 in H , then it is abelian. In this case $H^2 \subset \mathfrak{z}(H)$, forcing $H^3 = (0)$. This contradiction shows that $\mathfrak{z}(H)$ has codimension ≥ 3 in H . Since $H^3 \neq (0)$ has codimension 1 in H^2 , the equality $H^2 \cap \mathfrak{z}(H) = H^3$ holds. Therefore,

$$\begin{aligned} 3 &\leq \dim H/\mathfrak{z}(H) = \dim H/(H^2 + \mathfrak{z}(H)) + \dim H^2/H^3 \\ &\leq \dim H/(H^2 + \text{rad } L(\alpha)) + \dim H^2/H^3 = 3. \end{aligned}$$

This implies that $\mathfrak{z}(H)$ has codimension 3 in H .

Let $h \in \mathfrak{z}(H)$ and write $h = h_s + h_n$ with $h_s \in T$ and $h_n \in \mathcal{N}(H_p)$. In view of our earlier remarks, $h_n \in \mathfrak{z}(H) \cap (T + H)$. Because $\Gamma(L, T) = \Omega$, Theorem 3.3 shows that for every $\gamma \in \Gamma(L, T)$ the element $\Psi_\gamma(h_n) \in \Psi_\gamma(T) + \Psi_\gamma(H) = \Psi_\gamma(H)$ of $L[\gamma] \cong H(2; \underline{1})^{(2)} \oplus F(1 + x_1)^4 \partial_2$ is p -nilpotent in $L[\gamma]$ and commutes with $\Psi_\alpha(H)$. Arguing as in the proof of Lemma 3.4 it is now straightforward to see that $\Psi_\gamma(h_n) = 0$. Then $[h_n, L(\gamma)] \subset \text{rad } L(\gamma)$. In view of Theorem 4.7, this entails that $[h_n, L_\gamma] = 0$ for all $\gamma \in \Gamma(L, T)$. As a consequence, $h_n = 0$, forcing $\mathfrak{z}(H) = H \cap T$. Combined with our remarks in part (b) this gives (iii), completing the proof. \square

Corollary 5.6. *Let $\alpha, \beta \in \Gamma(L, T)$. Then case 3) of Proposition 4.2 does not occur for $L[\alpha, \beta]$.*

Proof. Indeed, otherwise the T -socle of $L[\alpha, \beta]$ has the form $S_1 \oplus S_2 = S_1(\delta_1) \oplus S_2(\delta_2)$. Then $\Psi_{\alpha, \beta}(H) \cap S_i(\delta_i) \cong \Psi_{\delta_i}(H)$ for $i = 1, 2$. As $\delta_1, \delta_2 \in \Omega$ by Corollary 5.3(i), it follows from Theorem 3.3 that $S_i(\delta_i) \cong H(2; \underline{1})^{(2)} \oplus F(1 + x_2)^4 \partial_2$ and $\Psi_{\delta_i}(H)$ is a nonabelian Cartan subalgebra of $S_i(\delta_i)$. Then Lemma 3.4 implies that $\dim \Psi_{\delta_i}(H^2) = 2$. As a consequence, $\Psi_{\alpha, \beta}(H^2) \cap S_i(\delta_i)$ is 2-dimensional for $i = 1, 2$. But then $\dim H^2 \geq 4$ contrary to Theorem 5.5(ii). The result follows. \square

Corollary 5.7. *The following are true:*

- (1) $\Gamma(L, T) \cup \{0\}$ is an \mathbb{F}_p -subspace of T^* .
- (2) The p -envelope of H^3 in L_p coincides with T .
- (3) $H_p = H + T$.

Proof. (1) Since every $\gamma \in \Gamma(L, T)$ is Hamiltonian by Theorem 4.7, we have $\mathbb{F}_p^\times \gamma \subset \Gamma(L, T)$. Let $\alpha, \beta \in \Gamma(L, T)$ be \mathbb{F}_p -independent. Then $\Gamma(L[\alpha, \beta], \Psi_{\alpha, \beta}(T))$ contains two nonsolvable roots. In view of Corollary 5.6, this implies that $L[\alpha, \beta]$ is determined by cases 5) or 6) of Proposition 4.2. In both cases, $\Gamma(L[\alpha, \beta], \Psi_{\alpha, \beta}(T)) \cup \{0\} = \mathbb{F}_p \alpha + \mathbb{F}_p \beta$; see Lemma 2.5(4) and [P 94, Lemmas 4.1 & 4.4]. As a consequence, $\alpha + \beta \in \Gamma(L, T)$. Statement (1) follows.

(2) By Theorem 5.5(3), $H^3 \subset T$. Denote by T_0 the p -envelope of H^3 in T and suppose that $T_0 \neq T$. Then T_0 is a proper subtorus of T . By part (1), there exists $\gamma \in \Gamma(L, T)$ such that $\gamma(T_0) = 0$. Then $\gamma(H^3) = 0$ contrary to Corollary 5.3(i). Therefore, $(H^3)_p = T$.

(3) It is immediate from Theorem 5.5(i) that $H_p \subset H + T$. Since $T = (H^3)_p \subset H_p$ by part (2), we now derive that $H_p = H + T$. \square

We now summarize the results of this section:

Theorem 5.8. *Let L, T and H be as above. Then the following hold:*

- 1) $\Gamma(L, T) \cup \{0\}$ is an \mathbb{F}_p -subspace of T^* and no root in $\Gamma(L, T)$ vanishes on H^3 .
- 2) $H^3 \subset T$, $\mathfrak{z}(H) = H \cap T$, $H_p = H + T$, $\dim H/(H \cap T) = 3$, $\dim H^2 = 3$, and $\dim H^3 = 2$. The p -envelope of H^3 in L_p coincides with T .
- 3) $\text{rad } L(\alpha) = H \cap T \cap \ker \alpha$, $\dim L_\alpha = 5$, and $L[\alpha] \cong H(2; \underline{1})^{(2)} \oplus F(1 + x_1)^4 \partial_2$ for every $\alpha \in \Gamma(L, T)$.
- 4) If $\alpha, \beta \in \Gamma(L, T)$ are \mathbb{F}_p -independent, then either $L[\alpha, \beta] \cong \mathcal{M}(1, 1)$ or

$$H(2; (2, 1))^{(2)} \subset L[\alpha, \beta] \subset H(2; (2, 1)).$$

Furthermore, $L[\alpha, \beta] \cong L(\alpha, \beta)/H \cap T \cap \ker \alpha \cap \ker \beta$.

Proof. Parts 1) and 2) are just reformulations of our earlier results. In order to get 3) and 4) it suffices to observe that $\text{rad } L(\alpha) \subset \mathfrak{z}(H) = H \cap T$; see Corollary 5.4 and Theorem 5.5(iv). \square

6. Some properties of the restricted Melikian algebra

In order to proceed further with our investigation, we now need more information on central extensions and irreducible representations of the Melikian algebra $\mathcal{M}(1, 1)$.

Proposition 6.1. *Every Melikian algebra $\mathcal{M}(\underline{n})$, where $\underline{n} = (n_1, n_2)$, possesses a nondegenerate invariant symmetric bilinear form.*

Proof. Adopt the notation of [St 04, Sect. 4.3] and consider the natural grading

$$\mathcal{M}(\underline{n}) = \mathcal{M}_{-3} \oplus \mathcal{M}_{-2} \oplus \mathcal{M}_{-1} \oplus \mathcal{M}_0 \oplus \mathcal{M}_1 \oplus \cdots \oplus \mathcal{M}_s, \quad s = 3(5^{n_1} + 5^{n_2}) - 7$$

of the Melikian algebra $\mathcal{M} = \mathcal{M}(\underline{n})$. Recall that $\mathcal{M}_0 = \bigoplus_{i,j=1}^2 x_i \partial_j \cong \mathfrak{gl}(2)$, $\mathcal{M}_{-3} = F\partial_1 \oplus F\partial_2$ and $\mathcal{M}_s = Fx^{(\tau(\underline{n}))}\tilde{\partial}_1 \oplus Fx^{(\tau(\underline{n}))}\tilde{\partial}_2$, where $\tau(\underline{n}) = (5^{n_1} - 1, 5^{n_2} - 1)$. Both \mathcal{M}_{-3} and \mathcal{M}_s are 2-dimensional irreducible \mathcal{M}_0 -modules. Using the multiplication table [St 04, (4.3.1)], it is easy to observe that

$$\begin{aligned} [x_1 \partial_1, x^{(\tau(\underline{n}))}\tilde{\partial}_1] &= (-2 + 2)x^{(\tau(\underline{n}))}\tilde{\partial}_1 = 0, & [x_2 \partial_1, x^{(\tau(\underline{n}))}\tilde{\partial}_1] &= 0, \\ [x_2 \partial_2, x^{(\tau(\underline{n}))}\tilde{\partial}_1] &= (-1 + 2)x^{(\tau(\underline{n}))}\tilde{\partial}_1. \end{aligned}$$

This shows that $x^{(\tau(\underline{n}))}\tilde{\partial}_1$ is a primitive vector of weight $(0, 1)$ for the Borel subalgebra $\mathfrak{b} := Fx_1 \partial_1 \oplus Fx_2 \partial_2 \oplus Fx_2 \partial_1$ of \mathcal{M}_0 . Now let f be the linear function on \mathcal{M}_{-3} such that $f(\partial_1) = 0$ and $f(\partial_2) = 1$. Then $(x_1 \partial_1)(f) = -f \circ (x_1 \partial_1) = 0$, $(x_2 \partial_2)(f) = -f \circ (x_2 \partial_2) = f$ and $(x_2 \partial_1)(f) = -f \circ (x_2 \partial_1) = 0$, showing that $f \in (\mathcal{M}_{-3})^*$ is a primitive vector of weight $(0, 1)$ for the Borel subalgebra \mathfrak{b} . From this it is immediate that $(\mathcal{M}_{-3})^* \cong \mathcal{M}_s$ as \mathcal{M}_0 -modules. As \mathcal{M} is an irreducible graded \mathcal{M}_p -module, [P 85, Lemma 4] shows that there exists a module isomorphism $\theta: \mathcal{M} \xrightarrow{\sim} \mathcal{M}^*$ sending \mathcal{M}_i onto $(\mathcal{M}_{s-3-i})^*$ for all $i \in \{-3, \dots, s\}$ (as usual, we identify $(\mathcal{M}_i)^*$ with the subspace of \mathcal{M}^* consisting of all linear functions vanishing on all \mathcal{M}_k with $k \neq i$).

Define a bilinear form $b: \mathcal{M} \times \mathcal{M} \rightarrow F$ by setting $b(x, y) := (\theta(x))(y)$ for all $x, y \in \mathcal{M}$. Since θ is an isomorphism of \mathcal{M} -modules, the form b is nondegenerate and \mathcal{M} -invariant. Next we define a bilinear skew-symmetric form b' on \mathcal{M} by setting $b'(x, y) := b(x, y) - b(y, x)$ for all $x, y \in \mathcal{M}$. As \mathcal{M} is a simple Lie algebra, the invariant form b' is either nondegenerate or zero. As $\dim \mathcal{M} = 5^{n_1+n_2+1}$ is odd, it must be that $b' = 0$. Therefore, the form b is symmetric. \square

From now on we denote by \mathcal{M} the restricted Melikian algebra $\mathcal{M}(1, 1)$.

Proposition 6.2. *If $\tilde{\mathcal{M}}$ is a Lie algebra with center $\mathfrak{z} = \mathfrak{z}(\tilde{\mathcal{M}})$ such that $\tilde{\mathcal{M}}/\mathfrak{z} \cong \mathcal{M}$, then $\tilde{\mathcal{M}}^{(1)} \cong \mathcal{M}$ and $\tilde{\mathcal{M}} = \tilde{\mathcal{M}}^{(1)} \oplus \mathfrak{z}$.*

Proof. We need to show that the second cohomology group $H^2(\mathcal{M}, F)$ vanishes. Let b be the nondegenerate bilinear form from the proof of Proposition 6.1. By a standard argument explained in detail in [P 94, p. 681], for every 2-cocycle $\varphi: \mathcal{M} \times \mathcal{M} \rightarrow F$ there exists a derivation $d \in \text{Der } \mathcal{M}$ such that $b(d(x), y) = -b(x, d(y))$ and $\varphi(x, y) = b(d(x), y)$ for all $x, y \in \mathcal{M}$. Moreover, φ is a 2-coboundary if and only if the derivation d is inner. Since $\text{Der } \mathcal{M} = \text{ad } \mathcal{M}$ by [St 04, Thm. 7.1.4], for instance, we now obtain $H^2(\mathcal{M}, F) = 0$, as desired. \square

If V is an irreducible module over a finite dimensional restricted Lie algebra \mathcal{L} over F , then there exists a linear function $\chi = \chi_V \in \mathcal{L}^*$ such that for every $x \in \mathcal{L}$ the

central element $x^p - x^{[p]}$ of $U(\mathcal{L})$ acts on V as the scalar operator $\chi(x)^p \text{Id}_V$. The linear function χ is called the p -character of V . Given $f \in \mathcal{L}^*$ we denote by $\mathfrak{z}_{\mathcal{L}}(f)$ the stabilizer of f in \mathcal{L} . Recall that $\mathfrak{z}_{\mathcal{L}}(f) = \{x \in \mathcal{L} \mid f([x, \mathcal{L}]) = 0\}$ is a restricted subalgebra of even codimension in \mathcal{L} .

For our constructions in the final sections of this work we need some information on the p -characters of irreducible representations of dimension ≤ 125 of the restricted Melikian algebra $\mathcal{M} = \bigoplus_{i=-3}^s \mathcal{M}_i$.

Proposition 6.3. *If V is an irreducible \mathcal{M} -module of dimension ≤ 125 , then the p -character of V vanishes on the subspace $\bigoplus_{i \geq -2} \mathcal{M}_i$. If V has a nonzero p -character, then $\dim V = 125$.*

Proof. Write $\mathcal{M}^* = \bigoplus_{i=-3}^s (\mathcal{M}_i)^*$, where $(\mathcal{M}_i)^* = \{f \in \mathcal{M}^* \mid \bigoplus_{j \neq i} \mathcal{M}_j \subset \ker f\}$ and $s = 3(5 + 5) - 7 = 23$. Let χ be the p -character of the \mathcal{M} -module V . If $\chi = 0$, then there is nothing to prove; so suppose $\chi \neq 0$. Then $\chi = \sum_{i=-3}^d \chi_i$, where $\chi_i \in (\mathcal{M}_i)^*$ and $\chi_d \neq 0$.

(a) We first suppose that $d > 0$ and let $2q = \text{codim}_{\mathcal{M}} \mathfrak{z}_{\mathcal{M}}(\chi_d)$. Then [P-Sk 99, Prop. 5.5] yields that $5^q \mid \dim V$. Since $\dim V \leq 5^3$, it follows that $\mathfrak{z}_{\mathcal{M}}(\chi_d)$ has codimension ≤ 6 in \mathcal{M} . Let b be the \mathcal{M} -invariant nondegenerate bilinear form from the proof of Proposition 6.1. Then $\chi_d = b(z, \cdot) = \theta(z)$ for some nonzero $z \in \mathcal{M}_{s-3-d}$ and $\mathfrak{z}_{\mathcal{M}}(\chi_d) = \mathfrak{c}_{\mathcal{M}}(z)$. It follows that the set

$$\mathcal{X} := \{x \in \mathcal{M}_{s-3-d} \mid \text{codim}_{\mathcal{M}} \mathfrak{c}_{\mathcal{M}}(x) \leq 6\}$$

is nonzero. It is straightforward to see that \mathcal{X} is a Zariski closed, conical subset of \mathcal{M}_{s-3-d} invariant under the subgroup $\text{Aut}_0 \mathcal{M}$ of all automorphisms of \mathcal{M} preserving the natural grading of \mathcal{M} . Let $\mathbb{P}(\mathcal{X})$ be the closed subset of the projective space $\mathbb{P}(\mathcal{M}_{s-3-d})$ corresponding to \mathcal{X} and let \mathbf{T} denote the 2-dimensional torus of the algebraic group $\text{Aut}_0 \mathcal{M}$ whose group of rational characters is described in [Sk 01, p. 72]. Note that the Lie algebra of \mathbf{T} equals $F(\text{ad } x_1 \partial_1) \oplus F(\text{ad } x_2 \partial_2)$.

The connected abelian group \mathbf{T} acts regularly on \mathcal{X} , hence fixes a point in $\mathbb{P}(\mathcal{X})$ by Borel's theorem. This means that there exists a nonzero $x_0 \in \mathcal{M}_{s-3-d}$ such that $\mathfrak{c}_{\mathcal{M}}(x_0)$ has codimension ≤ 6 in \mathcal{M} and $\mathbf{T} \cdot x_0 \subset Fx_0$. Let \mathfrak{n}_0 denote the normalizer of Fx_0 in \mathcal{M} and set $\mathfrak{t} := F(x_1 \partial_1) \oplus F(x_2 \partial_2)$, a 2-dimensional torus in \mathcal{M} . By our choice of x_0 (and \mathbf{T}) we have that $[\mathfrak{t}, x_0] \subset Fx_0$.

Suppose $[\mathfrak{t}, x_0] \neq 0$. Then $\mathfrak{n}_0 \supsetneq \mathfrak{c}_{\mathcal{M}}(x_0)$. As a consequence, \mathfrak{n}_0 is a proper subalgebra of codimension ≤ 5 in \mathcal{M} . By a result of Kuznetsov [Kuz 91, Thm. 4.7], every proper subalgebra of \mathcal{M} has codimension ≥ 5 and every subalgebra of codimension 5 contains $\bigoplus_{i \geq 1} \mathcal{M}_i$ (see also [St 04, Thm. 4.3.3] and [Sk 01, Sect. 1]). Since the subalgebra $\bigoplus_{i \geq 1} \mathcal{M}_i$ of \mathfrak{n}_0 acts nilpotently on \mathcal{M} , it must annihilate Fx_0 . On the other hand, it is immediate from the simplicity of the graded Lie algebra \mathcal{M} that the graded subspace $\text{Ann}_{\mathcal{M}}(\bigoplus_{i > 0} \mathcal{M}_i)$ coincides with \mathcal{M}_s . So $x_0 \in \mathcal{M}_s$ forcing $d = -3$, a contradiction.

Now suppose $[\mathfrak{t}, x_0] = 0$. Using [St 04, (4.3.1)] one checks immediately that $\mathfrak{c}_{\mathcal{M}}(\mathfrak{t}) = \mathfrak{t} \oplus Fx_1^3 x_2^3 \oplus Fx_1^4 x_2^3 \tilde{\partial}_1 \oplus Fx_1^3 x_2^4 \tilde{\partial}_2$. In view of [St 04, p. 200], we have that $\mathfrak{t} \subset \mathcal{M}_0$, $x_1^2 x_2^2 \in \mathcal{M}_{10}$ and $Fx_1^4 x_2^3 \tilde{\partial}_1 \oplus Fx_1^3 x_2^4 \tilde{\partial}_2 \subset \mathcal{M}_{20}$. As $d > 0$ by our present assumption, we have $s - 3 - d = 23 - 3 - d < 20$. Rescaling x_0 if need be we thus may assume that either $x_0 = x_1^2 x_2^2$ or $x_0 = x_1 \partial_1 + \alpha x_2 \partial_2$ for some $\alpha \in F$ (by symmetry). Applying [St 04, (4.3.1)] it is easy to observe that $\mathfrak{c}_{\mathcal{M}_i}(x_1^2 x_2^2) = (0)$ for $i < 0$ and $\mathfrak{c}_{\mathcal{M}_0}(x_1^2 x_2^2) = \mathfrak{t}$.

This shows that the case $x_0 = x_1^2 x_2^2$ is impossible (as $\mathfrak{c}_{\mathcal{M}}(x_0)$ has codimension ≤ 6 in \mathcal{M}). If $x_0 = x_1 \partial_1 + \alpha x_2 \partial_2$, then $[x_0, \mathcal{M}]$ contains all $x_1^i \partial_1$ with $i \in \{0, 2, 3, 4\}$ and all $x_1^j x_2 \partial_2$ with $j \in \{1, 2, 3, 4\}$. It follows that $\text{codim}_{\mathcal{M}} \mathfrak{c}_{\mathcal{M}}(x_0) \geq 8$ in this case, showing that the case where $d > 0$ cannot occur.

(b) Thus $d \leq 0$. Recall from [Sk 01, p. 72] that the group of rational characters of \mathbf{T} has \mathbb{Z} -basis $\{\varepsilon_1, \varepsilon_2\}$ and the \mathbf{T} -weight vectors $\partial_1, \partial_2 \in \mathcal{M}_{-3}$, $1 \in \mathcal{M}_{-2}$, $\tilde{\partial}_1, \tilde{\partial}_2 \in \mathcal{M}_{-1}$ and $x_1 \partial_2, x_2 \partial_1 \in \mathcal{M}_0$ have weights $-2\varepsilon_1 - \varepsilon_2, -\varepsilon_1 - 2\varepsilon_2, -\varepsilon_1 - \varepsilon_2, -\varepsilon_1, -\varepsilon_2$ and $\varepsilon_1 - \varepsilon_2, -\varepsilon_1 + \varepsilon_2$, respectively.

Assume that $\chi_0(x_1 \partial_2) \neq 0$ and consider the cocharacter $\varepsilon_1^*: F^\times \rightarrow \text{Aut } \mathcal{M}$ such that $(\varepsilon_1^*(t))(x) = t^n x$ for all $t \in F^\times$ and all weight vectors $x \in \mathcal{M}_{n\varepsilon_1 + m\varepsilon_2}$, where $m, n \in \mathbb{Z}$. Let $\mathcal{M} = \bigoplus_{i \in \mathbb{Z}} \mathcal{M}(i)$ be the \mathbb{Z} -grading of \mathcal{M} induced by ε_1^* . Since $d \leq 0$ and $\chi_0(x_1 \partial_2) \neq 0$ by our assumption, we have that $\chi = \chi(-2) + \chi(-1) + \chi(0) + \chi(1)$, where $\chi(i) \in \mathcal{M}(i)^*$ and $\chi(1) \neq 0$. Applying [P-Sk 99, Prop. 5.5] to the graded Lie algebra $\bigoplus_{i \in \mathbb{Z}} \mathcal{M}(i)$ we deduce that $\mathfrak{z}_{\mathcal{M}}(\chi(1))$ has codimension ≤ 6 in \mathcal{M} . Since in the present case $x_1 \partial_1 \in \mathfrak{n}_{\mathcal{M}}(F\chi(1)) \setminus \mathfrak{z}_{\mathcal{M}}(\chi(1))$, the normalizer $\mathfrak{n}_{\mathcal{M}}(F\chi(1))$ has codimension ≤ 5 in \mathcal{M} . Using Kuznetsov's description of subalgebras of codimension 5 in \mathcal{M} and arguing as in part (a) we now obtain that $\chi(1) = b(y, \cdot)$ for some $y \in \mathcal{M}_s$. Since in the present case $s - 3 - d \neq s$, we reach a contradiction, thereby showing that $\chi_0(x_1 \partial_2) = 0$. Arguing in a similar fashion one obtains that χ_0 vanishes on $x_2 \partial_1$.

(c) Thus we may assume from now that $d \leq 0$ and χ_0 vanishes on $F(x_1 \partial_2) \oplus F(x_2 \partial_1)$. In this situation [P-Sk 99, Prop. 5.5] is no longer useful, so we have to argue differently. Denote by \mathfrak{g} the Lie subalgebra of \mathcal{M} generated by the graded components $\mathcal{M}_{\pm 1}$. Using [St 04, (4.3.1)] it is easy to check that $\mathcal{M}_1 = Fx_1 \oplus Fx_2$, $\mathcal{M}_1^2 = F(x_1 \tilde{\partial}_1 + x_2 \tilde{\partial}_2)$, $\mathcal{M}_1^3 = F(x_1^2 \partial_1 + x_1 x_2 \partial_2) \oplus F(x_1 x_2 \partial_1 + x_2^2 \partial_2)$ and $\mathcal{M}_1^4 = (0)$. Then it is immediate from [St 04, Thm. 5.4.1] that \mathfrak{g} is a 14-dimensional simple Lie algebra of type G_2 . We identify χ with its restriction to \mathfrak{g} , denote by \mathbf{G} the simple algebraic group $\text{Aut } \mathfrak{g}$, and regard $\mathbf{L} := \text{Aut}_0 \mathcal{M}$ as a Levi subgroup of \mathbf{G} . Clearly, \mathbf{T} is a maximal torus of \mathbf{G} contained in \mathbf{L} . Also, $\text{Lie}(\mathbf{G}) = \text{ad } \mathfrak{g}$ and 5 is a good prime for the root system $\Phi = \Phi(\mathbf{G}, \mathbf{T})$. Since the Killing form κ of the Lie algebra \mathfrak{g} is nondegenerate, we may identify \mathfrak{g} with \mathfrak{g}^* via the \mathbf{G} -equivariant map sending $x \in \mathfrak{g}$ to the linear function $\kappa(x, \cdot) \in \mathfrak{g}^*$.

Let \mathbf{P} be the parabolic subgroup of \mathbf{G} with $\text{Lie}(\mathbf{P}) = \text{ad}(\bigoplus_{i \geq 0} \mathfrak{g}_i)$, where $\mathfrak{g}_i = \mathfrak{g} \cap \mathcal{M}_i$, and let Φ^+ be a positive system in Φ containing the \mathbf{T} -weights of $\bigoplus_{i > 0} \mathfrak{g}_i$. Let $\{\alpha_1, \alpha_2\}$ be the basis of simple roots of Φ contained in Φ^+ . Adopting Bourbaki's numbering we will assume that \mathfrak{g}_0 is spanned by \mathfrak{t} and root vectors $e_{\pm \alpha_2}$ and \mathfrak{g}_1 is spanned by root vectors e_{α_1} and $e_{\alpha_1 + \alpha_2}$. We stress that α_1 is a *short* root of Φ .

Since $g(\chi_0) = \chi_0$ for all $g \in \mathbf{T}$ and $\chi_{-1} + \chi_{-2} + \chi_{-3}$ is a linear combination of \mathbf{T} -weight vectors corresponding to positive roots, the Zariski closure of $\mathbf{T} \cdot \chi$ contains χ_0 . It follows that $\dim \mathbf{G} \cdot \chi \geq \dim \mathbf{G} \cdot \chi_0$. Since χ_0 vanishes on all root vectors $e_\alpha \in \mathfrak{g}$ with $\alpha \in \Phi$ and 5 is a good prime for Φ , the stabilizer $Z_{\mathbf{G}}(\chi_0)$ of χ_0 in \mathbf{G} is a Levi subgroup of \mathbf{G} ; see [P 95, (3.1)] and references therein. Since the \mathfrak{g} -module V has p -character χ , the Kac–Weisfeiler conjecture proved in [P 95] shows that $5^{(\dim \mathbf{G} \cdot \chi)/2} \mid \dim V$.

Suppose $\chi_0 \neq 0$. Then $Z_{\mathbf{G}}(\chi_0)$ is a proper Levi subgroup of \mathbf{G} . Since any Levi subgroup of \mathbf{G} is conjugate to a standard Levi subgroup, this implies that

$\dim Z_{\mathbf{G}}(\chi_0) \leq 4$. As a consequence,

$$\dim \mathbf{G} \cdot \chi \geq \dim \mathbf{G} \cdot \chi_0 = \dim \mathbf{G} - \dim Z_{\mathbf{G}}(\chi_0) \geq 10.$$

But then $5^5 \mid \dim V$, a contradiction. Thus, $\chi = \kappa(y_1 + y_2 + y_3, \cdot)$ for some $y_i \in \mathfrak{g}_i$.

Suppose $y_1 \neq 0$. Since y is a nilpotent element of \mathfrak{g} , all nonzero scalar multiples of y are \mathbf{G} -conjugate. From this it is immediate that the Zariski closure of $\mathbf{G} \cdot y$ contains y_1 , implying $\dim \mathbf{G} \cdot y \geq \dim \mathbf{G} \cdot y_1$. As all nonzero elements of \mathfrak{g}_1 are conjugate under the action of \mathbf{L} , we may assume that $y_1 = e_{\alpha_1}$. As $\dim \mathfrak{c}_{\mathfrak{g}}(e_{\alpha_1}) = 6$, it follows that

$$\dim \mathbf{G} \cdot \chi = \dim \mathbf{G} \cdot y \geq \dim \mathbf{G} \cdot y_1 = \dim \mathbf{G} - \dim Z_{\mathbf{G}}(y_1) \geq \dim \mathbf{G} - \dim \mathfrak{c}_{\mathfrak{g}}(y_1) = 8.$$

Applying [P 95, Thm. I] now gives $5^4 \mid \dim V$. Since this is false, it must be that $y_1 = 0$. If $y_2 \neq 0$, then y_2 is a nonzero multiple of $e_{2\alpha_1+\alpha_2}$ (for $\mathfrak{g}_2 = [\mathcal{M}_1, \mathcal{M}_1] = Fe_{2\alpha_1+\alpha_2}$). As $y = y_2 + y_3$, it is easy to see that the orbit $\mathbf{P} \cdot y$ contains $e_{2\alpha_1+\alpha_2}$. As $2\alpha_1 + \alpha_2$ is a short root of Φ , we can argue as before to obtain $5^4 \mid \dim V$, a contradiction.

As a result, $y = y_3$. Then $\chi = \chi_{-3}$ vanishes on $\bigoplus_{i \geq -2} \mathcal{M}_i$ as stated. If $\chi \neq 0$, then we can assume that $y = e_{3\alpha_1+2\alpha_2}$ (for all nonzero elements in $\mathfrak{g}_3 = [e_{2\alpha_1+\alpha_2}, \mathcal{M}_1]$ are conjugate under the action of \mathbf{L}). Since $\dim \mathfrak{c}_{\mathfrak{g}}(e_{3\alpha_1+2\alpha_2}) = 8$, it follows from [P 95, Thm. I] that $5^3 \mid \dim V$. Then $\dim V = 125$, completing the proof. \square

7. Melikian pairs

Set $\Gamma := \Gamma(L, T)$. According to Theorem 5.8(4), if $\alpha, \beta \in \Gamma$ are \mathbb{F}_p -independent, then either $L[\alpha, \beta] \cong \mathcal{M}$ or $H(2; (2, 1))^{(2)} \subset L[\alpha, \beta] \subset H(2; (2, 1))$. If $L[\alpha, \beta] \cong \mathcal{M}$ we say that $(\alpha, \beta) \in \Gamma^2$ is a *Melikian pair*. Recall from Theorem 5.8(2) that H^3 is a 2-dimensional subspace of T .

Lemma 7.1. *A pair $(\alpha, \beta) \in \Gamma^2$ is Melikian if and only if $H^3 \cap \ker \alpha \neq H^3 \cap \ker \beta$, i.e. if and only if $\alpha|_{H^3}$ and $\beta|_{H^3}$ are linearly independent over F .*

Proof. Suppose $H(2; (2, 1))^{(2)} \subset L[\alpha, \beta] \subset H(2; (2, 1))$. Recall from Sect. 2 that $H(2; (2, 1)) = H(2; (2, 1))^{(2)} \oplus V$ and $V^3 = (0)$. Then $L[\alpha, \beta]^3 \subset H(2; (2, 1))^{(2)}$, forcing $\Psi_{\alpha, \beta}(H)^3 \subset H(2; (2, 1))^{(2)}$. But then $\Psi_{\alpha, \beta}(H^3) \subset \Psi_{\alpha, \beta}(T) \cap H(2; (2, 1))^{(2)}$ has dimension ≤ 1 by Lemma 2.4. In view of Theorem 5.8(4) and the inclusion $H^3 \subset T$, this means that $H^3 \cap \ker \alpha \cap \ker \beta$ has codimension ≥ 1 in H^3 . It follows that α and β are linearly dependent as linear functions on H^3 .

Now suppose that $L[\alpha, \beta] \cong \mathcal{M}$. In view of Theorem 5.8(1), both α and β are in Ω . Therefore, $\Psi_{\alpha, \beta}(T)$ is a nonstandard 2-dimensional torus in $L[\alpha, \beta] \cong \text{Der } L[\alpha, \beta]$. Applying [P 94, Lemmas 4.1 & 4.4] now gives $\dim \Psi_{\alpha, \beta}(H)^3 = 2$, which in conjunction with Theorem 5.8(5) yields that $H^3 \cap \ker \alpha \cap \ker \beta$ has codimension ≤ 2 in H^3 . So α and β must be linearly independent on H^3 . \square

Corollary 7.2. *For any $\alpha \in \Gamma$ there exists $\beta \in \Gamma$ such that (α, β) is a Melikian pair.*

Proof. It follows from Theorem 5.8 that $H^3 \cap \ker \alpha = Ft$ for some nonzero $t \in H^3$. Since $H^3 \subset T$ and L is centerless, there is a $\beta \in \Gamma$ with $\beta(t) \neq 0$. Then (α, β) is a Melikian pair by Lemma 7.1. \square

Lemma 7.3. *If (α, β) is a Melikian pair, then*

$$L_p(\alpha, \beta) = L(\alpha, \beta)^{(1)} \oplus T \cap \ker \alpha \cap \ker \beta, \quad L_p(\alpha, \beta)^{(1)} = L(\alpha, \beta)^{(1)} \cong \mathcal{M}.$$

Proof. (a) Since $\text{rad}_T L(\alpha, \beta) = H \cap T \cap \ker \alpha \cap \ker \beta$ by Theorem 5.8(5), we have that $\text{rad}_T L(\alpha, \beta) = \mathfrak{z}(L(\alpha, \beta))$. Hence

$$(0) \longrightarrow H \cap T \cap \ker \alpha \cap \ker \beta \longrightarrow L(\alpha, \beta) \longrightarrow \mathcal{M} \longrightarrow (0)$$

is a central extension \mathcal{M} . By Proposition 6.2, this extension splits; that is, $L(\alpha, \beta) = L(\alpha, \beta)^{(1)} \oplus H \cap T \cap \ker \alpha \cap \ker \beta$ and $L(\alpha, \beta)^{(1)} \cong \mathcal{M}$.

(b) Note that $L_p(\alpha, \beta) = \tilde{H} + L(\alpha, \beta)$, where $\tilde{H} = \mathfrak{c}_{L_p}(T)$, and $[\tilde{H}, L(\alpha, \beta)^{(1)}] \subset L(\alpha, \beta)^{(1)}$. Hence \tilde{H} acts on $L(\alpha, \beta)^{(1)}$ as derivations. As all derivations of $L(\alpha, \beta)^{(1)} \cong \mathcal{M}$ are inner by [St 04, Thm. 7.1.4], it must be that $\tilde{H} = H' \oplus \tilde{H}_0$, where $\tilde{H}_0 = \mathfrak{c}_{\tilde{H}}(L(\alpha, \beta)^{(1)})$ and $H' = L(\alpha, \beta)^{(1)} \cap H$. From part (a) of this proof it follows that $H \subset T + H'$. Consequently, $[H, \tilde{H}_0] = 0$.

Put $\Gamma' := \{\gamma \mid \gamma(H') \neq 0\}$ and let μ be any root in Γ' . Recall that $\dim L_\mu = 5$; see Theorem 5.8(3). As H' is a nontriangularable Cartan subalgebra of $L(\alpha, \beta)^{(1)} \cong \mathcal{M}$ by [P 94, Lemmas 4.1 & 4.4], the H' -module L_μ is irreducible. But then \tilde{H}_0 acts on L_μ as scalar operators. On the other hand, it follows from Schue's lemma [St 04, Prop. 1.3.6(1)] that L is generated by the root spaces L_γ with $\gamma \in \Gamma'$. It follows that \tilde{H}_0 acts semisimply on L , implying $\tilde{H}_0 \subset T$. From this it is immediate that $\tilde{H}_0 = T \cap \ker \alpha \cap \ker \beta$. As a result,

$$L_p(\alpha, \beta) = L(\alpha, \beta)^{(1)} + \tilde{H}_0 = L(\alpha, \beta)^{(1)} \oplus T \cap \ker \alpha \cap \ker \beta,$$

finishing the proof. \square

Let (α, β) be a Melikian pair. Note that $T_0 := T \cap \ker \alpha \cap \ker \beta$ is a restricted ideal of $L_p(\alpha, \beta)$ and $T = H^3 \oplus T_0$. So the Lie algebra $L_p(\alpha, \beta)/T_0$ inherits a p th power map from $L_p(\alpha, \beta)$. Since $L_p(\alpha, \beta)/T_0 \cong \mathcal{M}$ by Lemma 7.3 and both Lie algebras are centerless and restricted, every isomorphism between $L_p(\alpha, \beta)/T_0$ and \mathcal{M} is an isomorphism of restricted Lie algebras. Any such isomorphism maps the torus T/T_0 of the restricted Lie algebra $L_p(\alpha, \beta)/T_0$ onto a 2-dimensional nonstandard torus of \mathcal{M} . According to [P 94, Lemmas 4.1 & 4.4], any such torus is conjugate under $\text{Aut } \mathcal{M}$ to the torus $\mathfrak{t} := F(1 + x_1)\partial_1 \oplus F(1 + x_2)\partial_2$.

Recall from Sect. 6 the natural grading of the Lie algebra \mathcal{M} . For $i \geq -3$, we set $\mathcal{M}_{(i)} := \bigoplus_{j \geq i} \mathcal{M}_i$. The decreasing filtration $(\mathcal{M}_{(i)})_{i \geq -3}$ of the Lie algebra \mathcal{M} can be regarded as a standard (Weisfeiler) filtration of \mathcal{M} associated with its maximal subalgebra $\mathcal{M}_{(0)}$. It is referred to as the *natural* filtration of \mathcal{M} , because $\mathcal{M}_{(0)}$ is the only subalgebra of codimension 5 and depth 3 in \mathcal{M} . All components $\mathcal{M}_{(i)}$ of this filtration are invariant under the automorphism group of \mathcal{M} ; see [St 04, Thm. 4.3.3(2) and Rem. 4.3.4] for more detail. Note that $\mathcal{M} = \mathfrak{t} \oplus \mathcal{M}_{(-2)}$.

Regard $\tilde{\mathcal{M}} := \mathcal{M} \oplus T_0$ as a direct sum of Lie algebras and define a p th power map $u \mapsto u^p$ on $\tilde{\mathcal{M}}$ by setting $u^p = u^{[p]}$ for all $u \in \mathcal{M}$ and $u^p = 0$ for all $u \in T_0$ (here $u \mapsto u^{[p]}$ is the p th power map on \mathcal{M}). The above discussion in conjunction with Lemma 7.3 shows that there exists a Lie algebra isomorphism

$$(7.1) \quad \Phi: L_p(\alpha, \beta) \xrightarrow{\sim} \tilde{\mathcal{M}} = \mathcal{M}_{(-2)} \oplus \Phi(T)$$

such that

$$(7.2) \quad \Phi(L(\alpha, \beta)^{(1)}) = \mathcal{M}, \quad \Phi(H^3) = \mathfrak{t}, \quad \Phi|_{T_0} = \text{Id}_{T_0}.$$

Note that Φ maps $L_p(\alpha, \beta)^{(1)}$ onto $\tilde{\mathcal{M}}^{(1)} = \mathcal{M}$. We stress that H^3 is not a restricted subalgebra of $L_p(\alpha, \beta)$, whilst $\Phi(H^3)$ is a maximal torus of $\tilde{\mathcal{M}}$. There exists a p -linear mapping $\Lambda: \tilde{\mathcal{M}} \longrightarrow \mathfrak{z}(\tilde{\mathcal{M}}) = T_0$ such that

$$\Lambda(u) = \Phi^{-1}(u)^{[p]} - \Phi^{-1}(u^p) \quad (\forall u \in \tilde{\mathcal{M}}),$$

where $\Phi^{-1}(u) \longmapsto \Phi^{-1}(u)^{[p]}$ is the p th power map in L_p .

Lemma 7.4. *The p -linear mapping Λ vanishes on the subspace $\mathcal{M}_{(-2)}$ of $\tilde{\mathcal{M}}$.*

Proof. Suppose $\Lambda(u) \neq 0$ for some $u \in \mathcal{M}_{(-2)}$. Then there is $\gamma \in \Gamma$ which does not vanish on $\Lambda(u) \in T_0 \setminus \{0\}$. Since $\Lambda(u) \subset T \cap \ker \alpha \cap \ker \beta$, the root γ is \mathbb{F}_p -independent of α and β . Let $M(\gamma; \alpha, \beta) := \bigoplus_{i,j \in \mathbb{F}_p} L_{\gamma+i\alpha+j\beta}$. By Theorem 5.8, $M(\gamma; \alpha, \beta)$ is 125-dimensional submodule of the $(T + L(\alpha, \beta)_p)$ -module L . The map $\text{ad} \circ \Phi^{-1}$ gives $M(\gamma; \alpha, \beta)$ an \mathcal{M} -module structure. Note that T_0 acts on $M(\gamma; \alpha, \beta)$ as scalar operators. This means that the \mathcal{M} -module $M(\gamma; \alpha, \beta)$ has a p -character; we call it χ . It is straightforward to see that $\Lambda(x) = \chi(x)^p$ for all $x \in \mathcal{M}$. But then χ does not vanish on $\mathcal{M}_{(-2)}$. Since $\dim M(\gamma; \alpha, \beta) = 125$, this contradicts Proposition 6.3. The result follows. \square

We now set $(L_p(\alpha, \beta)^{(1)})_{(i)} := \Phi^{-1}(\mathcal{M}_{(i)})$ for all $i \geq -3$. Then the following hold:

- $(L_p(\alpha, \beta)^{(1)})_{(-3)} = L_p(\alpha, \beta)^{(1)}$;
- $(L_p(\alpha, \beta)^{(1)})_{(0)}$ is a subalgebra of codimension 5 in $L_p(\alpha, \beta)^{(1)}$;
- $u^{[p]} \in L_p(\alpha, \beta)^{(1)}$ for all $u \in (L_p(\alpha, \beta)^{(1)})_{(-2)}$;
- $(L_p(\alpha, \beta)^{(1)})_{(0)}$ is a restricted subalgebra of $L_p(\alpha, \beta)$.

Since the natural filtration of \mathcal{M} is invariant under all automorphisms of \mathcal{M} (see [St 04, Rem. 4.3.4(3)]), the above definition of the subspaces $(L_p(\alpha, \beta)^{(1)})_{(i)}$ is independent of the choice of Φ satisfying (7.1) and (7.2).

8. Describing $L_p(\alpha)$

Fix $\alpha \in \Gamma$ and pick $\beta \in \Gamma$ be such that (α, β) is a Melikian pair; see Corollary 7.2. As before, we put $T_0 := T \cap \ker \alpha \cap \ker \beta$ and let Φ be a map satisfying (7.1) and (7.2). It gives rise to the *restricted* Lie algebra isomorphism

$$\bar{\Phi}: L_p(\alpha, \beta)/T_0 \xrightarrow{\sim} \mathcal{M} = \mathcal{M}_{(-2)} \oplus \bar{\Phi}(H^3), \quad \bar{\Phi}(H^3) = \mathfrak{t}.$$

By Theorem 5.8(1), no root in Γ vanishes on H^3 . As $\dim H^3 = 2$, there exists a nonzero $h_\alpha \in H^3$ such that $Fh_\alpha = H^3 \cap \ker \alpha$. As $\bar{\Phi}(Fh_\alpha)$ is a 1-dimensional subtorus of the nonstandard torus \mathfrak{t} , it follows from [Sk 01, Thm. 2.1] that there is an automorphism of \mathcal{M} which maps \mathfrak{t} onto itself and $F\bar{\Phi}(h_\alpha)$ onto $F(1+x_1)\partial_1$. Hence we may assume without loss of generality that

$$(8.1) \quad \Phi(L_p(\alpha)) = \mathfrak{c}_{\mathcal{M}}((1+x_1)\partial_1) \oplus T_0, \quad \Phi(T) = \mathfrak{t} \oplus T_0, \quad \bar{\Phi}(h_\alpha) = (1+x_1)\partial_1.$$

For $f \in \mathcal{O}(2; (1,1))_{(0)}$ set $f^{(k)} := f^k/k!$ for $0 \leq k \leq 4$ and $f^{(k)} := 0$ for $k < 0$ and $k \geq 5$. Direct computations show that $\mathfrak{c}_{\mathcal{M}}((1+x_1)\partial_1)$ has basis

$$\{x_2^{(r)}\partial_2, x_2^{(r)}(1+x_1)\partial_1, x_2^{(r)}(1+x_1)^2, x_2^{(r)}(1+x_1)^3\tilde{\partial}_2, x_2^{(r)}(1+x_1)^4\tilde{\partial}_1 \mid 0 \leq r \leq 4\}.$$

Using the multiplication table in [St 04, (4.3.1)] it is easy to observe that

$$\begin{aligned}
[x_2^{(r)}\partial_2, x_2^{(s)}\partial_2] &= \left[\binom{r+s-1}{r} - \binom{r+s-1}{s}\right]x_2^{(r+s-1)}\partial_2; \\
[x_2^{(r)}(1+x_1)\partial_1, x_2^{(s)}\partial_2] &= -\binom{r+s-1}{s}x_2^{(r+s-1)}(1+x_1)\partial_1; \\
[x_2^{(r)}(1+x_1)\partial_1, x_2^{(s)}(1+x_1)\partial_1] &= 0; \\
[x_2^{(r)}(1+x_1)^2, x_2^{(s)}\partial_2] &= -\left[\binom{r+s-1}{s} - 2\binom{r+s-1}{s-1}\right]x_2^{(r+s-1)}(1+x_1)^2; \\
[x_2^{(r)}(1+x_1)^2, x_2^{(s)}(1+x_1)\partial_1] &= -\left[2\binom{r+s}{s} - 2\binom{r+s}{s}\right]x_2^{(r+s)}(1+x_1)^2 = 0; \\
[x_2^{(r)}(1+x_1)^2, x_2^{(s)}(1+x_1)^2] &= 2\left[-\binom{r+s-1}{r} + \binom{r+s-1}{s}\right]x_2^{(r+s-1)}(1+x_1)^4\tilde{\partial}_1; \\
[x_2^{(r)}(1+x_1)^3\tilde{\partial}_2, x_2^{(s)}\partial_2] &= -\binom{r+s}{r}x_2^{(r+s-1)}(1+x_1)^3\tilde{\partial}_2; \\
[x_2^{(r)}(1+x_1)^3\tilde{\partial}_2, x_2^{(s)}(1+x_1)\partial_1] &= \binom{r+s-1}{r}x_2^{(r+s-1)}(1+x_1)^4\tilde{\partial}_1; \\
[x_2^{(r)}(1+x_1)^3\tilde{\partial}_2, x_2^{(s)}(1+x_1)^2] &= -\binom{r+s}{r}x_2^{(r+s)}\partial_2; \\
[x_2^{(r)}(1+x_1)^3\tilde{\partial}_2, x_2^{(s)}(1+x_1)^3\tilde{\partial}_2] &= 0; \\
[x_2^{(r)}(1+x_1)^4\tilde{\partial}_1, x_2^{(s)}\partial_2] &= -\left[\binom{r+s-1}{s} + 2\binom{r+s-1}{s-1}\right]x_2^{(r+s-1)}(1+x_1)^4\tilde{\partial}_1; \\
[x_2^{(r)}(1+x_1)^4\tilde{\partial}_1, x_2^{(s)}(1+x_1)\partial_1] &= -\left[3\binom{r+s}{r} + 2\binom{r+s}{s}\right]x_2^{(r+s)}\tilde{\partial}_1 = 0; \\
[x_2^{(r)}(1+x_1)^4\tilde{\partial}_1, x_2^{(s)}(1+x_1)^2] &= -\binom{r+s}{r}x_2^{(r+s)}(1+x_1)\partial_1; \\
[x_2^{(r)}(1+x_1)^4\tilde{\partial}_1, x_2^{(s)}(1+x_1)^3\tilde{\partial}_2] &= \binom{r+s}{r}x_2^{(r+s)}(1+x_1)^2; \\
[x_2^{(r)}(1+x_1)^4\tilde{\partial}_1, x_2^{(s)}(1+x_1)^4\tilde{\partial}_1] &= 0.
\end{aligned}$$

In order to obtain a more invariant description of $L_p(\alpha)$ we now consider a vector space $R = R' \oplus C$ over F with $\dim C = \dim T - 2$ such that R' has basis $\{x_1^{(i)}x_2^{(j)} \mid 0 \leq i, j \leq 4, 1 \leq i+j \leq 7\} \cup \{x_2^{(5)}\} \cup \{z\}$. We give R a Lie algebra structure by setting

$$[x_1^{(i)}x_2^{(j)}, x_1^{(k)}x_2^{(l)}] := \left[\binom{i+k-1}{i-1}\binom{j+l-1}{j} - \binom{i+k-1}{i}\binom{j+l-1}{j-1}\right]x_1^{(i+k-1)}x_2^{(j+l-1)}$$

for all i, j, k, l with $3 \leq i+j+k+l \leq 7$ such that $(j, l) \neq (0, 0)$ whenever $i+k=5$, and by requiring that $[Fz+C, R] = 0$ and

$$[x_1^{(i)}x_2^{(j)}, x_1^{(k)}x_2^{(l)}] := \begin{cases} 0 & \text{if } i+j+k+l \leq 2, \\ (-1)^i z & \text{if } j=l=0 \text{ and } i+k=5. \end{cases}$$

The Lie algebra R is a (nonsplit) central extension of $H(2; \underline{1})^{(2)} \oplus FD_H(x_2^{(5)})$. Computations show that

$$\begin{aligned}
[x_1 x_2^{(r)}, x_1 x_2^{(s)}] &= \left[\binom{r+s-1}{r} - \binom{r+s-1}{s} \right] x_1 x_2^{(r+s-1)}; \\
[-x_1^{(4)} x_2^{(r-1)}, x_1 x_2^{(s)}] &= \begin{cases} -\binom{r+s-1}{s} (-x_1^{(4)} x_2^{(r+s-2)}) & \text{if } r+s \geq 2, \\ -z & \text{if } r=1, s=0; \end{cases} \\
[-x_1^{(4)} x_2^{(r-1)}, -x_1^{(4)} x_2^{(s-1)}] &= 0; \\
[x_1^{(2)} x_2^{(r)}, x_1 x_2^{(s)}] &= -\left[\binom{r+s-1}{s} - 2\binom{r+s-1}{s-1} \right] x_1^{(2)} x_2^{(r+s-1)}; \\
[x_1^{(2)} x_2^{(r)}, -x_1^{(4)} x_2^{(s-1)}] &= 0; \\
[x_1^{(2)} x_2^{(r)}, x_1^{(2)} x_2^{(s)}] &= 2\left[-\binom{r+s-1}{r} + \binom{r+s-1}{s} \right] x_1^{(3)} x_2^{(r+s-1)}; \\
[x_2^{(r+1)}, x_1 x_2^{(s)}] &= -\binom{r+s}{r} x_2^{(r+s)}; \\
[x_2^{(r+1)}, -x_1^{(4)} x_2^{(s-1)}] &= \binom{r+s-1}{r} x_1^{(3)} x_2^{(r+s-1)}; \\
[x_2^{(r+1)}, x_1^{(2)} x_2^{(s)}] &= -\binom{r+s}{r} x_1 x_2^{(r+s)}; \\
[x_2^{(r+1)}, x_2^{(s+1)}] &= 0; \\
[x_1^{(3)} x_2^{(r)}, x_1 x_2^{(s)}] &= -\left[\binom{r+s-1}{s} + 2\binom{r+s-1}{s-1} \right] x_1^{(3)} x_2^{(r+s-1)}; \\
[x_1^{(3)} x_2^{(r)}, -x_1^{(4)} x_2^{(s-1)}] &= 0; \\
[x_1^{(3)} x_2^{(r)}, x_1^{(2)} x_2^{(s)}] &= \begin{cases} -\binom{r+s}{r} (-x_1^{(4)} x_2^{(r+s-1)}) & \text{if } r+s \geq 1, \\ -z & \text{if } r=s=0; \end{cases} \\
[x_1^{(3)} x_2^{(r)}, x_2^{(s+1)}] &= \binom{r+s}{r} x_1^{(2)} x_2^{(r+s)} \\
[x_1^{(3)} x_2^{(r)}, x_1^{(3)} x_2^{(s)}] &= 0.
\end{aligned}$$

By comparing the displayed multiplications tables it is straightforward to see that the following statement holds:

Proposition 8.1. *Any linear map $\Theta': \mathfrak{c}_{\widetilde{\mathcal{M}}}((1+x_1)\partial_1) \longrightarrow R$ which takes T_0 isomorphically onto C and satisfies the conditions*

$$\begin{aligned}
\Theta'(x_2^{(r)}(1+x_1)\partial_1) &= \begin{cases} -x_1^{(4)} x_2^{(r-1)} & \text{if } 1 \leq r \leq 4, \\ z & \text{if } r=0, \end{cases} \\
\Theta'(x_2^{(r)}\partial_2) &= x_1 x_2^{(r)}, & 0 \leq r \leq 4, \\
\Theta'(x_2^{(r)}(1+x_1)^2) &= x_1^{(2)} x_2^{(r)}, & 0 \leq r \leq 4, \\
\Theta'(x_2^{(r)}(1+x_1)^3 \tilde{\partial}_2) &= x_2^{(r+1)}, & 0 \leq r \leq 4, \\
\Theta'(x_2^{(r)}(1+x_1)^4 \tilde{\partial}_1) &= x_1^{(3)} x_2^{(r)}, & 0 \leq r \leq 4,
\end{aligned}$$

is an isomorphism of Lie algebras.

We now fix Θ' described in Proposition 8.1 and set $\Theta := \Theta' \circ \Phi|_{L_p(\alpha)}$, where $\Phi: L_p(\alpha, \beta) \xrightarrow{\sim} \widetilde{\mathcal{M}}$ is a Lie algebra isomorphism satisfying (7.1), (7.2) and (8.1). Clearly, $\Theta: L_p(\alpha) \xrightarrow{\sim} R$ is a Lie algebra isomorphism. We give R a p th power map

by setting

$$(8.2) \quad r^p := \Theta(\Theta^{-1}(r)^{[p]}) \quad (\forall r \in R).$$

This turns Θ into an isomorphism of restricted Lie algebras. Because the p -linear map $\Lambda: \tilde{\mathcal{M}} \rightarrow T_0$ vanishes on the subspace $\mathcal{M}_{(-2)}$ of $\tilde{\mathcal{M}}$ by Lemma 7.4 and Θ is defined via Φ , the explicit description of Θ' in Proposition 8.1 shows that the map (8.2) has the following properties:

$$(8.3) \quad \begin{aligned} (x_2^{(r+1)})^p &= 0 & \text{if } 0 \leq r \leq 4; \\ (x_1 x_2^{(r)})^p &= 0 & \text{if } r \neq 0, 1; \\ (x_1^{(2)} x_2^{(r)})^p &= 0 & \text{if } 0 \leq r \leq 4; \\ (x_1^{(3)} x_2^{(r)})^p &= 0 & \text{if } 0 \leq r \leq 4; \\ (x_1^{(4)} x_2^{(r-1)})^p &= 0 & \text{if } 1 \leq r \leq 4; \\ (x_1 x_2)^p &= x_1 x_2 & \text{i.e. } x_1 x_2 \text{ is toral} \end{aligned}$$

(we refer to [Sk 01] for more detail on the p -structure in the restricted Melikian algebra). Note that $(x_1)^p$ and z^p lie in $\Theta(T) = Fz \oplus C$. Moreover, $Fz = \Theta(H^3 \cap \ker \alpha)$ coincides the image of $F(1 + x_1)\partial_1$ under Φ^{-1} and $\Theta'((1 + x_2)\partial_2) = x_1 + x_1 x_2$.

We stress that all constructions of Sections 7 and 8 depend on the choice of a Melikian pair.

9. The subalgebra $Q(\alpha)$

The results obtained so far apply to all nonstandard tori of maximal dimension in L_p . However, such tori need not be conjugate under the automorphism group of L . In order to identify L with one of the Melikian algebras, we will require a sufficiently generic nonstandard torus of maximal dimension in L_p .

Proposition 9.1. *There exists a nonstandard torus T' of maximal dimension in L_p for which $(\mathfrak{c}_L(T'))^3$ contains no nonzero toral elements of L_p .*

Proof. Let T and Γ be as Sect. 8 and let $(\alpha, \beta) \in \Gamma^2$ be a Melikian pair. Choose an isomorphism $\Phi: L_p(\alpha, \beta) \xrightarrow{\sim} \tilde{\mathcal{M}}$ satisfying (7.1) and (7.2). Then $H^3 = \Phi^{-1}(\mathfrak{t})$. Set $q_i := \Phi^{-1}(x_i \partial_i)$, $n_i = \Phi^{-1}(\partial_i)$ and $h_i := n_i^{[p]}$, where $i = 1, 2$. As the elements $x_i \partial_i$ are toral in \mathcal{M} , Lemma 7.4 says that both q_1 and q_2 are toral elements of L_p . Note that $T = F(q_1 + n_1) \oplus F(q_2 + n_2) \oplus T_0$, where $T_0 = T \cap \ker \alpha \cap \ker \beta$.

As Φ is a Lie algebra isomorphism, it is straightforward to see that $[q_i, n_i] = -n_i$ and $h_i \in T_0$ for $i = 1, 2$. So it follows from Jacobson's formula that $(q_i + n_i)^{[p]^k} = q_i + n_i + \sum_{j=0}^{k-1} h_i^{[p]^j}$ for all $k \geq 1$. Since $(H^3)_p = T$ by Theorem 5.8(3) and $H^3 = F(q_1 + n_1) \oplus F(q_2 + n_2)$, it follows that the p -closure of $Fh_1 + Fh_2$ coincides with T_0 .

Recall that $\dim T_0 \geq 1$. Let $\{t_1, \dots, t_s\}$ be a basis of T_0 consisting of toral elements of L_p . For $x = \sum_{j=1}^s \alpha_j t_j \in T_0$ define $\text{Supp}(x) := \{j \mid \alpha_j \neq 0\}$. Write $h_1 = \sum_{j=1}^s \lambda_j t_j$ and $h_2 = \sum_{j=1}^s \mu_j t_j$ with $\lambda_j, \mu_j \in F$. Since the $[p]$ -th powers of h_1 and h_2 span T_0 , it must be that

$$\text{Supp}(h_1) \cup \text{Supp}(h_2) = \{1, \dots, s\}.$$

In particular, $h_1 \neq 0$ or $h_2 \neq 0$. Recall from Sect. 6 the maximal torus \mathbf{T} of the group $\text{Aut}_0 \mathcal{M}$ of all automorphisms of \mathcal{M} preserving the natural grading of \mathcal{M} . For every $\sigma \in \text{Aut}_0 \mathcal{M}$ the subalgebra $\Phi^{-1}(\sigma(\mathbf{t}) + T_0)$ is a nonstandard torus of maximal dimension in L_p and the elements $(\Phi^{-1} \circ \sigma)(x_1 \partial_1)$ and $(\Phi^{-1} \circ \sigma)(x_2 \partial_2)$ are toral in L_p by Lemma 7.4. Since the group $\text{Aut}_0 \mathcal{M}$ acts transitively on the set of bases of \mathcal{M}_{-3} , there is $\tau \in \text{Aut}_0 \mathcal{M}$ such that the elements $((\Phi^{-1} \circ \tau)(\partial_1))^{[p]}$ and $((\Phi^{-1} \circ \tau)(\partial_2))^{[p]}$ are both nonzero. Replacing \mathbf{t} by $\tau(\mathbf{t})$ and renumbering the t_i 's if necessary, we thus may assume that λ_1 and μ_1 are both nonzero.

Since F is infinite, there exist $a, b \in F^\times$ such that the elements $a^p \lambda_1$ and $b^p \mu_1$ of F are linearly independent over \mathbb{F}_p . Applying a suitable automorphism from the subgroup \mathbf{T} of $\text{Aut}_0 \mathcal{M}$ one observes that $\mathbf{t}' := F(a + x_1) \partial_1 \oplus F(b + x_2) \partial_2$, is a 2-dimensional nonstandard torus in \mathcal{M} and $\mathbf{t}' = (\mathbf{c}_\mathcal{M}(\mathbf{t}'))^3$ (alternatively, one can apply [P 94, Lemmas 4.1 & 4.4]). This entails that

$$T' := \Phi^{-1}(\mathbf{t}' \oplus T_0) = F(q_1 + an_1) \oplus F(q_2 + bn_2) \oplus T_0$$

is a nonstandard torus of maximal dimension in L_p with $F(q_1 + an_1) \oplus F(q_2 + bn_2) = (\mathbf{c}_L(T'))^3$. Suppose

$$(9.1) \quad (x(q_1 + an_1) + y(q_2 + bn_2))^{[p]} = x(q_1 + an_1) + y(q_2 + bn_2)$$

for some $x, y \in F$. Applying Φ to both sides of (9.1) gives

$$(x(a + x_1) \partial_1 + y(b + x_2) \partial_2)^{[p]} = x(a + x_1) \partial_1 + y(b + x_2) \partial_2.$$

As both $(a + x_1) \partial_1$ and $(b + x_2) \partial_2$ are toral elements of \mathcal{M} , we get $x, y \in \mathbb{F}_p$. Hence

$$\begin{aligned} x(q_1 + an_1) + y(q_2 + bn_2) &= (x(q_1 + an_1) + y(q_2 + bn_2))^{[p]} \\ &= x(q_1 + an_1 + a^p h_1) + y(q_2 + bn_2 + b^p h_2), \end{aligned}$$

implying $xa^p h_1 + yb^p h_2 = 0$. As a consequence, $xa^p \lambda_j + yb^p \mu_j = 0$ for all $j \leq s$. But then $a^p \lambda_1$ and $b^p \mu_1$ are linearly dependent over \mathbb{F}_p , a contradiction. We conclude that $(\mathbf{c}_L(T'))^3$ contains no nonzero toral elements of L_p . \square

Retain the notation introduced in Sections 7 and 8. In view of Proposition 9.1, we may assume that for every $\alpha \in \Gamma$ no nonzero element of $H^3 \cap \ker \alpha$ is toral in L_p .

The map $\Theta: L_p(\alpha) \xrightarrow{\sim} R$ defined in Sect. 8 induces a natural Lie algebra isomorphism

$$\bar{\Theta}: L_p(\alpha)/\mathfrak{z}(L_p(\alpha)) \xrightarrow{\sim} R/\mathfrak{z}(R) \cong H(2; \underline{1})^{(2)} \oplus FD_H(x_2^{(5)}).$$

Let $(R/\mathfrak{z}(R))_{(i)}$ denote the i th component of the standard filtration of the Cartan type Lie algebra $R/\mathfrak{z}(R)$, where $i \geq -1$, and denote by $L_p(\alpha)_{(i)}$ the inverse image of $(R/\mathfrak{z}(R))_{(i)}$ under $\bar{\Theta}$. We thus obtain a filtration $\{L_p(\alpha)_{(i)} \mid i \geq -1\}$ of the Lie algebra $L_p(\alpha)$ with $\bigcap_{i \geq -1} L_p(\alpha)_{(i)} = T \cap \ker \alpha$ and $\dim(L_p(\alpha)/L_p(\alpha)_{(0)}) = 2$. This filtration is, in fact, independent of the choice of $\bar{\Theta}$, because $(R/\mathfrak{z}(R))_{(0)}$ is the unique subalgebra of codimension 2 in the Cartan type Lie algebra $R/\mathfrak{z}(R)$. Since $\bar{\Theta}$ is a restricted Lie algebra isomorphism, all $L_p(\alpha)_{(i)}$ are restricted subalgebras of $L_p(\alpha)$. We denote by $\text{nil}_{[p]}(L_p(\alpha)_{(i)})$ the maximal ideal of $L_p(\alpha)_{(i)}$ consisting of p -nilpotent elements of L_p .

Definition 9.1. Define

$$\begin{aligned} W &:= \{u \in L_p(\alpha)^{(1)} \cap L_p(\alpha)_{(0)} \mid u^{[p]} \in L_p(\alpha)^{(1)}\}; \\ P &:= \{u \in W \mid [u, W] \subset W\}; \\ Q(\alpha) &:= P + \text{nil}_{[p]}(L_p(\alpha)_{(3)}). \end{aligned}$$

Because of the uniqueness of the filtration $\{L_p(\alpha)_{(i)} \mid i \geq -1\}$ this definition is independent of the choices made earlier. The main result of this section is the following:

Proposition 9.2. *If (α, β) is a Melikian pair in Γ^2 , then*

$$Q(\alpha) = L_p(\alpha) \cap (L_p(\alpha, \beta)^{(1)})_{(0)}.$$

Proof. (a) Choose any Lie algebra isomorphism $\Phi: L_p(\alpha, \beta) \xrightarrow{\sim} \tilde{\mathcal{M}} = \mathcal{M} \oplus T_0$ satisfying (7.1), (7.2) and (8.1). Then $\Phi(L_p(\alpha) \cap (L_p(\alpha, \beta)^{(1)})_{(0)})$ is spanned by

$$\{x_2^{(r)}\partial_2, x_2^{(r)}(1+x_1)\partial_1, x_2^{(r)}(1+x_1)^2, x_2^{(r)}(1+x_1)^3\tilde{\partial}_2, x_2^{(r)}(1+x_1)^4\tilde{\partial}_1 \mid 1 \leq r \leq 4\}.$$

Let $\Theta = \Phi \circ \Theta': L_p(\alpha) \xrightarrow{\sim} R$ be the isomorphism associated with Φ . The explicit formulae for Θ' yield that $\Theta(L_p(\alpha) \cap (L_p(\alpha, \beta)^{(1)})_{(0)})$ is spanned by the set

$$\{x_1x_2^{(r)}, x_1^{(2)}x_2^{(r)}, x_1^{(3)}x_2^{(r)} \mid 1 \leq r \leq 4\} \cup \{x_1^{(4)}x_2^{(r)} \mid 0 \leq r \leq 3\} \cup \{x_2^{(r)} \mid 2 \leq r \leq 5\};$$

see Proposition 8.1.

(b) Next we are going to determine $\Theta(W)$, $\Theta(P)$ and $\Theta(Q(\alpha))$ by using Definition 9.1. First we observe that

$$\Theta(L_p(\alpha)^{(1)} \cap L_p(\alpha)_{(0)}) = Fz \oplus (\bigoplus_{0 \leq i, j \leq 4, 2 \leq i+j \leq 7} Fx_1^{(i)}x_2^{(j)});$$

see Proposition 8.1. It is immediate from equations (8.3) that

$$(x_1^{(i)}x_2^{(j)})^p \in \Theta(L_p(\alpha)^{(1)} \cap L_p(\alpha)_{(0)}) \quad \text{whenever } i+j \geq 2.$$

Recall that Θ is an isomorphism of restricted Lie algebras. In conjunction with Jacobson's formula, this shows that $\Theta(W)$ is a subspace of R . As a consequence, we have the inclusion

$$\bigoplus_{0 \leq i, j \leq 4, 2 \leq i+j \leq 7} Fx_1^{(i)}x_2^{(j)} \subset \Theta(W).$$

On the other hand, if $z \in \Theta(W)$, then the definition of Θ' and our assumption on Φ yield $H^3 \cap \ker \alpha \subset W$. Then $h_\alpha \in W$. As $Fh_\alpha = H^3 \cap \ker \alpha = F\Theta^{-1}(z)$, our assumption on h_α in (8.1) yields $h_\alpha = \Phi^{-1}((1+x_1)\partial_1)$. It follows that $h_\alpha^{[p]} - h_\alpha \in L_p(\alpha)^{(1)} \cap T_0$. As $h_\alpha^{[p]} \neq h_\alpha$ by our choice of T , this entails $L_p(\alpha, \beta)^{(1)} \cap T_0 \neq (0)$ contradicting Lemma 7.3. We conclude that

$$\Theta(W) = \bigoplus_{0 \leq i, j \leq 4, 2 \leq i+j \leq 7} Fx_1^{(i)}x_2^{(j)}.$$

Let $u = \sum_{i,j} s_{i,j} x_1^{(i)}x_2^{(j)} \in \Theta(P)$. Since $x_1^{(2)}, x_1^{(3)} \in \Theta(W)$ and $[x_1^{(2)}, x_1^{(3)}] = z$, it follows readily from the definition of P that $s_{2,0} = s_{3,0} = 0$. The multiplication table for R given Sect. 8 now shows that $\Theta(P)$ is spanned by

$$\{x_1^{(4)}, x_2^{(2)}, x_2^{(3)}\} \cup \{x_1^{(i)}x_2, x_1^{(i)}x_2^{(2)}, x_1^{(i)}x_2^{(3)} \mid 1 \leq i \leq 4\} \cup \{x_1^{(i)}x_2^{(4)} \mid 0 \leq i \leq 3\}.$$

(c) Finally, the nilpotent subalgebra $\Theta(L_p(\alpha)_{(3)})$ is spanned by

$$\{x_1^{(i)}x_2^{(4)} \mid 0 \leq i, j \leq 4; 5 \leq i+j \leq 7\} \cup \{x_2^{(5)}, z\} \cup C.$$

By (8.3), the Lie product of any two elements in this set is p -nilpotent in R . Since Θ is an isomorphism of restricted Lie algebras, it follows that $\Theta(\text{nil}_{[p]}(L_p(\alpha)_{(3)}))$ is spanned by $\{x_1^{(i)}x_2^{(4)} \mid 0 \leq i, j \leq 4; 5 \leq i+j \leq 7\} \cup \{x_2^{(5)}\}$. Comparing the spanning set of $\Theta(L_p(\alpha) \cap (L_p(\alpha, \beta)^{(1)})_{(0)})$ from part (a) of this proof with that of $\Theta(Q(\alpha)) = \Theta(P) + \Theta(\text{nil}_{[p]}(L_p(\alpha)_{(3)}))$ we now obtain that

$$\Theta(L_p(\alpha) \cap (L_p(\alpha, \beta)^{(1)})_{(0)}) = \Theta(Q(\alpha)).$$

Since Θ is an isomorphism, the proposition follows. \square

Remark 9.3. Proposition 9.2 implies that $Q(\alpha)$ is a subalgebra of $L(\alpha)$.

At the end of Sect. 8 we mentioned that $\Theta'((1+x_2)\partial_2) = x_1(1+x_2)$. In what follows we require some computations in the subalgebra $\Theta(H) \subset \mathfrak{c}_R(x_1(1+x_2))$. It follows from the multiplication table for R that $\mathfrak{c}_R(x_1(1+x_2))$ contains $x_1^{(2)}(1+x_2)^2$ and $x_1^{(3)}(1+x_2)^3$. Set $w := x_2 - x_2^{(2)} + 2x_2^{(3)} - x_2^{(4)} - x_2^{(5)}$ and observe that

$$(9.2) \quad [x_1(1+x_2), w] = [x_1, w] + [x_1x_2, w] = (-x_2 + 2x_2^{(2)} - x_2^{(3)} - x_2^{(4)}) \\ + \left(x_2 - \binom{2}{1}x_2^{(2)} + 2\binom{3}{1}x_2^{(3)} - \binom{4}{1}x_2^{(4)} \right) = 0.$$

Applying Proposition 8.1 it is now easy to see that

$$\bigoplus_{i=1}^3 Fx_1^{(i)}(1+x_2)^i \oplus Fw \oplus Fz \subset \Theta(L_p(\alpha, \beta)^{(1)} \cap H_p) \subset \Theta(H).$$

Direct computations show that

$$(9.3) \quad [x_1^{(2)}(1+x_2)^2, w] = x_1(1+x_2)^2(1-x_2+2x_2^{(2)}-x_2^{(3)}-x_2^{(4)}) \\ = x_1(1+x_2)^6 = x_1(1+x_2);$$

$$(9.4) \quad [x_1^{(3)}(1+x_2)^3, w] = x_1^{(2)}(1+x_2)^3(1-x_2+2x_2^{(2)}-x_2^{(3)}-x_2^{(4)}) \\ = x_1^{(2)}(1+x_2)^7 = x_1^{(2)}(1+x_2)^2.$$

Proposition 9.4. *Let α be an arbitrary root of Γ . Then for any $r \in \mathbb{F}_p^\times$ there exists a linear map $l_{r\alpha}: L_{r\alpha} \rightarrow H$ such that $x - l_{r\alpha}(x) \in Q(\alpha)$ for all $x \in L_{r\alpha}$. Furthermore, $H \cap Q(\alpha) = (0)$ and $L(\alpha) = H + Q(\alpha)$.*

Proof. In order to perform computations in $L_p(\alpha)$ we are going to invoke the isomorphism $\Theta = \Theta' \circ \Phi$; see Proposition 8.1. Recall that

$$\Theta(T) = Fx_1(1+x_2) \oplus Fz \oplus C.$$

Replacing α by an \mathbb{F}_p^\times -multiple of α , if necessary, we may assume that $\alpha((x_1(1+x_2))) = 1$. Using the multiplication table for R it is then straightforward to see that

$$\Theta(L_{r\alpha}) = \bigoplus_{i=1}^3 Fx_1^{(i)}(1+x_2)^{r+i} \oplus F(x_1^{(4)}(1+x_2)^{r-1} - r^{-1}z) \oplus F((1+x_2)^r - 1)$$

for all $r \in \mathbb{F}_p^\times$ and that $\Theta(H)$ is sandwiched between $\bigoplus_{i=1}^3 Fx_1^{(i)}(1+x_2)^i \oplus Fw \oplus Fz$ and $\Theta(H_p) = \bigoplus_{i=1}^3 Fx_1^{(i)}(1+x_2)^i \oplus Fw \oplus Fz \oplus C$. We now define a linear map

$l_{r\alpha}: L_{r\alpha} \rightarrow H$ by the formula $l_{r\alpha} = \Theta^{-1} \circ m_r \circ \Theta$, where m_r is the linear map from $\Theta(L_{r\alpha})$ into $\Theta(H)$ given by

$$\begin{aligned} m_r(x_1^{(4)}(1+x_2)^{r-1} - r^{-1}z) &= -r^{-1}z; \\ m_r(x_1^{(i)}(1+x_2)^{r+i}) &= x_1^{(i)}(1+x_2)^i, \quad 1 \leq i \leq 3; \\ m_r((1+x_2)^r - 1) &= rw. \end{aligned}$$

Using the spanning set of $\Theta(Q(\alpha))$ from the proof of Proposition 9.2 one observes that $w - x_2 \in \Theta(Q(\alpha))$ and $x_1^{(i)}(1+x_2)^i - x_1^{(i)} \in \Theta(Q(\alpha))$ for $1 \leq i \leq 3$. By the same token, one finds that the subspace $\bigoplus_{i=1}^3 Fx_1^{(i)} \oplus Fx_2 \oplus Fz \oplus C$ of R complements $\Theta(Q(\alpha))$. Since $x_1^{(4)}(1+x_2)^{r-1} \in \Theta(Q(\alpha))$ for all $r \in \mathbb{F}_p^\times$, this implies that $y - m_r(y) \in \Theta(Q(\alpha))$ for all $y \in \Theta(L_{r\alpha})$ and $R = \Theta(H_p) \oplus \Theta(Q(\alpha))$.

As a result, $x - l_{r\alpha}(x) \in Q(\alpha)$ for all $r \in \mathbb{F}_p^\times$ and all $x \in L_{r\alpha}$. Consequently, $L_p(\alpha) = H_p \oplus Q(\alpha)$. Since $Q(\alpha) \subset L(\alpha)$, this yields $L(\alpha) = H \oplus Q(\alpha)$ and the proposition follows. \square

Proposition 9.5. *Let $\mathcal{N}(H)$ denote the set of all p -nilpotent elements of L_p contained in H . Then the following hold:*

- (1) $\mathcal{N}(H)$ is a 3-dimensional subspace of H .
- (2) There exists a unique 2-dimensional subspace $H_{(-1)}$ in $\mathcal{N}(H)$ satisfying the condition $[H_{(-1)}, H_{(-1)}] \subset \mathcal{N}(H)$. Moreover, $[H_{(-1)}, [H_{(-1)}, H_{(-1)}]] = H^3$.
- (3) For every $\alpha \in \Gamma$ the subspace $H_{(-1)} + Q(\alpha)$ is stable under the adjoint action of $Q(\alpha)$.

Proof. Jacobson's formula together with (8.3) and the multiplication table for R shows that the subspace $N := Fx_1^{(2)}(1+x_2)^2 \oplus Fx_1^{(3)}(1+x_2)^3 \oplus Fw$ consists of p -nilpotent elements of R . On the other hand, it is clear from our remarks in the proof of Proposition 9.4 that $\Theta(H_p) = \Theta(T) \oplus N$. Since $\Theta(T)$ is a torus, this entails that N coincides with the set of all p -nilpotent elements of the restricted Lie algebra $\Theta(H_p)$. Since $\Theta: L_p(\alpha) \xrightarrow{\sim} R$ is an isomorphism of restricted Lie algebras, we deduce that $\mathcal{N}(H) = \Theta^{-1}(N)$ is a 3-dimensional subspace of H .

The elements $D_H(x_1^{(2)}(1+x_2)^2)$ and $D_H(x_1^{(3)}(1+x_2)^3)$ of the Hamiltonian algebra $H(2; \underline{1})^{(2)}$ commute. Therefore, in our central extension R we have the equality

$$(9.5) \quad [x_1^{(2)}(1+x_2)^2, x_1^{(3)}(1+x_2)^3] = [x_1^{(2)}, x_2^{(3)}] = z.$$

Now take any linearly independent elements $u_1 = a_1x_1^{(2)}(1+x_2)^2 + b_1x_1^{(3)}(1+x_2)^3 + c_1w$ and $u_2 = a_2x_1^{(2)}(1+x_2)^2 + b_2x_1^{(3)}(1+x_2)^3 + c_2w$ in N such that $[u_1, u_2] \in N$. Then (9.5) together with (9.3) and (9.4) yields

$$N \ni [u_1, u_2] = (a_1b_2 - a_2b_1)z + (a_1c_2 - a_2c_1)x_1(1+x_2) + (b_1c_2 - b_2c_1)x_1^{(2)}(1+x_2)^2,$$

forcing $a_1b_2 = a_2b_1$ and $a_1c_2 = a_2c_1$. If $a_1 \neq 0$, then $u_2 = \frac{a_2}{a_1}u_1$ which is false. Therefore, $a_1 = 0$. Arguing similarly, one obtains $a_2 = 0$. This shows that $H_{(-1)} := \Theta^{-1}(Fx_1^{(3)}(1+x_2)^3 \oplus Fw)$ is the only 2-dimensional subspace of $\mathcal{N}(H)$ with the property that $[H_{(-1)}, H_{(-1)}] \subset \mathcal{N}(H)$. Combining (9.4), (9.3) and (9.5) one derives that $[H_{(-1)}, [H_{(-1)}, H_{(-1)}]] = H^3$.

Using the spanning set for $\Theta(Q(\alpha))$ displayed in part (a) the proof of Proposition 9.2 and the multiplication table for R , it is routine to check that

$$[\Theta(Q(\alpha)), Fx_1^{(3)}(1+x_2)^3 \oplus Fw] \subset \Theta(Q(\alpha)) + Fx_1^{(3)}(1+x_2)^3 \oplus Fw.$$

This implies that $H_{(-1)} + Q(\alpha)$ is invariant under the adjoint action of $Q(\alpha)$. \square

10. Conclusion

For any $\gamma \in \Gamma$ we fix a map $l_\gamma: L_\gamma \rightarrow H$ satisfying the conditions of Proposition 9.4. Given $x \in L_\gamma$ we set $\tilde{x} := x - l_\gamma(x)$, an element of $Q(\alpha)$. Define

$$L_{(0)} := \sum_{\gamma \in \Gamma} Q(\gamma),$$

a subspace of L . We are going to show that $L_{(0)}$ is actually a subalgebra of L . Since it follows from Remark 9.3 that $[Q(\gamma), Q(\gamma)] \subset L_{(0)}$ for all $\gamma \in \Gamma$, we just need to check that $[Q(\alpha), Q(\beta)] \subset L_{(0)}$ for all \mathbb{F}_p -independent $\alpha, \beta \in \Gamma$.

Lemma 10.1. *Let (α, β) be an arbitrary Melikian pair in Γ^2 and let $x \in L_\alpha$, $y \in L_\beta$. Then $[\tilde{x}, \tilde{y}] \in L_{(0)}$ and*

$$[\tilde{x}, \tilde{y}] \equiv \widetilde{[x, y]} \pmod{Q(\alpha) + Q(\beta)}.$$

Proof. Set $\Delta := \{\alpha\} \cup (\beta + \mathbb{F}_p\alpha)$. Proposition 9.4 says that $L(\delta) = H \oplus Q(\delta)$ for any $\delta \in \Delta$. In conjunction with Proposition 9.2, this gives

$$(10.1) \quad (L_p(\alpha, \beta)^{(1)})(\delta) = (H \cap L_p(\alpha, \beta)^{(1)}) \oplus Q(\delta) \quad (\forall \delta \in \Delta).$$

Recall that $\Phi: L_p(\alpha, \beta)^{(1)} \xrightarrow{\sim} \mathcal{M}$ is a Lie algebra isomorphism taking $H \cap L_p(\alpha, \beta)^{(1)}$ onto $\mathfrak{c}_\mathcal{M}(\mathfrak{t})$ and $(L_p(\alpha, \beta)^{(1)})_{(0)}$ onto $\mathcal{M}_{(0)}$. Therefore,

$$(10.2) \quad \dim H \cap L_p(\alpha, \beta)^{(1)} = 5, \quad \dim (L_p(\alpha, \beta)^{(1)})_{(0)} = 120.$$

Combining (10.2) and (10.1) we now deduce that for every $\delta \in \Delta$ the subalgebra $Q(\delta) = L_p(\delta) \cap (L_p(\alpha, \beta)^{(1)})_{(0)}$ has codimension 5 in the 1-section $(L_p(\alpha, \beta)^{(1)})(\delta)$. Since $L_p(\alpha, \beta)^{(1)} \cong \mathcal{M}$, it follows from [P 94, Lemmas 4.1 & 4.4], for instance, that $\dim (L_p(\alpha, \beta)^{(1)})(\delta) = 25$. Therefore, $\dim Q(\delta) = 20$ for all $\delta \in \Delta$.

For any $\mu \in \Delta$ one has

$$Q(\mu) \cap (\sum_{\delta \in \Delta \setminus \{\mu\}} Q(\delta)) \subset Q(\mu) \cap (\sum_{\delta \in \Delta \setminus \{\mu\}} L(\delta)) \subset Q(\mu) \cap H = (0).$$

This shows that the sum $Q(\alpha) + \sum_{j=0}^4 Q(\beta + j\alpha)$ is direct. But then

$$\dim \left(Q(\alpha) \oplus \bigoplus_{j=0}^4 Q(\beta + j\alpha) \right) = 6 \cdot 20 = 120 = \dim (L_p(\alpha, \beta)^{(1)})_{(0)},$$

implying that $(L_p(\alpha, \beta)^{(1)})_{(0)} = Q(\alpha) + \sum_{j \in \mathbb{F}_p} Q(\beta + j\alpha)$. As a consequence,

$$(10.3) \quad \begin{aligned} [Q(\alpha), Q(\beta)] &\subset \left[(L_p(\alpha, \beta)^{(1)})_{(0)}, (L_p(\alpha, \beta)^{(1)})_{(0)} \right] \subset (L_p(\alpha, \beta)^{(1)})_{(0)} \\ &= Q(\alpha) + \bigoplus_{j=0}^4 Q(\beta + j\alpha) \subset L_{(0)}. \end{aligned}$$

This shows that $[\tilde{x}, \tilde{y}] \in L_{(0)}$. Computing modulo $Q(\alpha) + Q(\beta)$ we get

$$\begin{aligned} [\tilde{x}, \tilde{y}] &= ([x, y] - l_{\alpha+\beta}([x, y] - [x, l_\beta(y)] + l_\alpha([x, l_\beta(y)] - [l_\alpha(x), y] + l_\beta([l_\alpha(x), y]) \\ &\quad + [l_\alpha(x), l_\beta(y)] + (l_{\alpha+\beta}([x, y]) - l_\alpha([x, l_\beta(y)]) - l_\beta([l_\alpha(x), y]))) \\ &= \widetilde{[x, y]} - \widetilde{[x, l_\beta(y)]} - \widetilde{[l_\alpha(x), y]} + \tilde{h} \\ &\equiv \widetilde{[x, y]} + \tilde{h}, \end{aligned}$$

where $\tilde{h} = l_{\alpha+\beta}([x, y]) - l_\alpha([x, l_\beta(y)]) - l_\beta([l_\alpha(x), y]) + [l_\alpha(x), l_\beta(y)]$. As $\widetilde{[x, y]} \in L_{(0)}$, it must be that $\tilde{h} \in H \cap L_{(0)} = H \cap (\sum_{\gamma \in \Gamma} Q(\gamma))$. Expressing $\tilde{h} = \sum_{\gamma \in \Gamma} (v_\gamma - l_\gamma(v))$ with $v_\gamma \in L_\gamma$ we see that $v_\gamma = 0$ for all γ , whence $l_\gamma(v_\gamma) = 0$ and $\tilde{h} = 0$. The result follows. \square

Theorem 10.2. $L_{(0)}$ is a proper subalgebra of L .

Proof. By our earlier remark in this section, we need to show that $[Q(\alpha), Q(\beta)] \subset L_{(0)}$ for all pairs $(\alpha, \beta) \in \Gamma^2$ such that α and β are \mathbb{F}_p -independent. If (α, β) is a Melikian pair, this follows from Lemma 10.1.

Take any \mathbb{F}_p -independent $\alpha, \beta \in \Gamma$ for which (α, β) is not a Melikian pair. Then $H^3 \cap \ker \alpha = H^3 \cap \ker \beta$; see Lemma 7.1. Recall that $H^3 \cap \ker \alpha = Fh_\alpha$ for some nonzero $h_\alpha \in H^3$. Put $\Gamma(\alpha) := \{\gamma \in \Gamma \mid \gamma(h_\alpha) \neq 0\}$. Since $H^3 \subset T$, the set $\Gamma(\alpha)$ is nonempty. Then it follows from Schue's lemma [St 04, Prop. 1.3.6(1)] that

$$(10.4) \quad L_\beta = \sum_{\gamma \in \Gamma(\alpha)} [L_\gamma, L_{\beta-\gamma}].$$

Let γ be an arbitrary root in $\Gamma(\alpha)$. Since $\alpha(h_\alpha) = \beta(h_\alpha) = 0$, it is immediate from Lemma 7.1 that (α, γ) and $(\alpha, \beta - \gamma)$ are Melikian pairs in Γ^2 .

Suppose $(\alpha + \gamma, \beta - \gamma)$ is not a Melikian pair. Then $(\beta - \gamma)(h_{\alpha+\gamma}) = 0$ by Lemma 7.1. As $(\beta - \gamma)(h_\alpha) = -\gamma(h_\alpha) \neq 0$ and $\dim H^3 = 2$ by Theorem 5.8(2), this yields $H^3 = Fh_\alpha \oplus Fh_{\alpha+\gamma}$. Also, $(\alpha + \beta)(h_\alpha) = 0$ and $(\alpha + \beta)(h_{\alpha+\gamma}) = ((\alpha + \gamma) + (\beta - \gamma))(h_{\alpha+\gamma}) = 0$ by our assumption on $(\alpha + \gamma, \beta - \gamma)$. This shows that $\alpha + \beta$ vanishes on H^3 and hence on $(H^3)_p = T$; see Theorem 5.8(2). But then $\alpha + \beta = 0$, a contradiction. Thus, $(\alpha + \gamma, \beta - \gamma)$ is a Melikian pair.

If $(\gamma, \alpha + \beta - \gamma)$ is not a Melikian pair, then $\gamma(h_{\alpha+\beta-\gamma}) = 0$. As $\gamma \in \Gamma(\alpha)$, we then have $H^3 = Fh_\alpha \oplus Fh_{\alpha+\beta-\gamma}$. But then $\alpha + \beta = \gamma + (\alpha + \beta - \gamma)$ vanishes on $(H^3)_p$, a contradiction. So $(\gamma, \alpha + \beta - \gamma)$ is a Melikian pair, too.

We now take arbitrary $u \in L_\alpha$ and $v \in L_\beta$. By (10.4), there exist $\gamma_1, \dots, \gamma_N \in \Gamma(\alpha)$ such that $v = \sum_{i=1}^N [x_i, y_i]$ for some $x_i \in L_{\gamma_i}$ and $y_i \in L_{\beta-\gamma_i}$, where $1 \leq i \leq N$. Applying Lemma 10.1 and the preceding remarks we obtain

$$\begin{aligned} [\tilde{u}, \tilde{v}] &\in \sum_{i=1}^N [\tilde{u}, [\tilde{x}_i, \tilde{y}_i]] + \sum_{i=1}^N [Q(\alpha), Q(\gamma_i) + Q(\beta - \gamma_i)] \\ &\subset \sum_{i=1}^N ([\tilde{u}, \tilde{x}_i], \tilde{y}_i] + [\tilde{x}_i, [\tilde{u}, \tilde{y}_i]]) + L_{(0)} \\ &\subset \sum_{i=1}^N ([Q(\alpha + \gamma_i), Q(\beta - \gamma_i)] + [Q(\gamma_i), Q(\alpha + \beta - \gamma_i)]) + L_{(0)} \subset L_{(0)}. \end{aligned}$$

Consequently, $[Q(\alpha), Q(\beta)] \subset L_{(0)}$ in all cases. The argument at the end of the proof of Lemma 10.1 shows that $L_{(0)} \cap H = (0)$. Hence $L_{(0)}$ is a proper subalgebra of L . \square

Recall the subspace $H_{(-1)}$ from Proposition 9.5(2). According to Proposition 9.5(3), $[Q(\gamma), H_{(-1)}] \subset H_{(-1)} + Q(\gamma) \subset H_{(-1)} + L_{(0)}$ for all $\gamma \in \Gamma$. In view of Theorem 10.4,

this means that

$$[L_{(0)}, H_{(-1)} + L_{(0)}] = [\sum_{\gamma \in \Gamma} Q(\gamma), H_{(-1)} + \sum_{\delta \in \Gamma} Q(\delta)] \subset H_{(-1)} + L_{(0)}.$$

Thus, $L_{(-1)} := H_{(-1)} + L_{(0)}$ is stable under the adjoint action of the subalgebra $L_{(0)}$.

We have finally come to the end of this tale. Let L' denote the subalgebra of L generated by $L_{(-1)}$. Proposition 9.5(2) shows that $H^3 \subset L'$. Then the p -envelope of L' in L_p contains $(H^3)_p = T$; see Theorem 5.8(2). As a consequence, L' is T -stable. Let γ be any root in Γ . Then $[T, x - l_\gamma(x)] \subset L'$ for all $x \in L_\gamma$, implying $L_\gamma \subset L'$. As this holds for all $\gamma \in \Gamma$ and L is simple, we deduce that $L' = L$.

It follows from Theorem 10.4 that $L_{(-1)} \supsetneq L_{(0)}$. We now consider the standard filtration of L associated with the pair $(L_{(-1)}, L_{(0)})$ (it is defined recursively by setting $L_{(i)} := \{x \in L_{(i-1)} \mid [x, L_{(i-1)}] \subset L_{(i-1)}\}$ and $L_{(-i)} := [L_{(-1)}, L_{(-i+1)}] + L_{(-i+1)}$ for all $i > 0$). Since L is simple and finite-dimensional, this filtration is exhaustive and separating. Let $G = \bigoplus_{i \in \mathbb{Z}} G_i$ denote the associated graded Lie algebra, where $G_i = \text{gr}_i(L) = L_{(i)}/L_{(i+1)}$.

Since $L_{(-1)} = H_{(-1)} + L_{(0)}$, we have that $L_{(-i)} = L_{(0)} + \sum_{j=1}^i (H_{(-1)})^j$ for all $i > 0$. Since $(H_{(-1)})^3 \subset H^3 \subset \mathfrak{z}(H)$ by Theorem 5.8(2), this shows that $L_{(-4)} = L_{(-3)}$, i.e. $G_{-4} = (0)$. As $\dim H_{(-1)} = 2$, we obtain by the same token that $\dim G_{-2} \leq 1$ and $\dim G_{-3} \leq 2$.

Let (α, β) be any Melikian pair in Γ^2 . By our remarks in the proof of Lemma 10.1, $(L_p(\alpha, \beta)^{(1)}) \cap L_{(0)} = (L_p(\alpha, \beta)^{(1)})_{(0)}$, while from the explicit description of $\Theta(H_{(-1)})$ in the proof of Proposition 9.5 and Proposition 8.1 we see that

$$(10.5) \quad H_{(-1)} + (L_p(\alpha, \beta)^{(1)})_{(0)} = (L_p(\alpha, \beta)^{(1)})_{(-1)}.$$

In particular, $H_{(-1)} \subset L_p(\alpha, \beta)^{(1)}$. It follows that the filtration of $L_p(\alpha, \beta)^{(1)} \cong \mathcal{M}$ induced by that of L has the property that

$$L_{(i)} = (L_p(\alpha, \beta)^{(1)} \cap L_{(i)}) + L_{(i-1)}, \quad i = -1, -2, -3.$$

In view of (10.5), this entails that $\dim G_{-1} = \dim G_{-3} = 2$ and $\dim G_2 = 1$.

As $\dim G_{-1} = 2$, and G_0 acts faithfully on G_{-1} , we have an embedding $G_0 \subset \mathfrak{gl}(2)$. As $(L_p(\alpha, \beta)^{(1)})_{(0)}$ acts on $(L_p(\alpha, \beta)^{(1)})_{(-1)}/(L_p(\alpha, \beta)^{(1)})_{(0)}$ as $\mathfrak{gl}(2)$, it follows from (10.5) that $(L_{(0)} \cap L_p(\alpha, \beta)^{(1)})/(L_{(1)} \cap L_p(\alpha, \beta)^{(1)}) \cong \mathfrak{gl}(2)$. As a consequence, $G_0 \cong \mathfrak{gl}(2)$. Finally, (10.5) yields that $L_p(\alpha, \beta)^{(1)} \cap L_{(4)} \neq (0)$, giving $G_4 \neq (0)$.

Applying [St 04, Thm. 5.4.1] we now obtain that the graded Lie algebra G is isomorphic to a Melikian algebra $\mathcal{M}(m, n)$ regarded with its natural grading. By a result of Kuznetsov [Kuz 91], any filtered deformation of the naturally graded Lie algebra $\mathcal{M}(m, n)$ is isomorphic to $\mathcal{M}(m, n)$; see [St 04, Thm. 6.7.3]. Thus, $L \cong \mathcal{M}(m, n)$, completing the proof of Theorem 1.2.

Corollary 10.3. *Let L be a finite-dimensional simple Lie algebra of Cartan type over an algebraically closed field of characteristic $p > 3$ and let T be a torus of maximal dimension in $L_p \subset \text{Der } L$. Then the centralizer of T in L_p acts triangulably on L .*

Proof. This is an immediate consequence of [P-St 04, Thm. A] and Theorem 1.2. \square

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